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Interpolation Spaces

An Introduction

With 5 Figures

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Preface

The works of Jaak Peetre constitute the main body of this treatise. Important contributors are also J.L. Lions and A.P. Calderón, not to mention several others. We, the present authors, have thus merely compiled and explained the works of others (with the exception of a few minor contributions of our own).

Let us mention the origin of this treatise. A couple of years ago, J. Peetre suggested to the second author, J. Löfström, writing a book on interpolation theory and he most generously put at Löfström's disposal an unfinished manuscript, covering parts of Chapter 1—3 and 5 of this book. Subsequently, Löfström prepared a first rough, but relatively complete manuscript of lecture notes. This was then partly rewritten and thouroughly revised by the first author, J. Bergh, who also prepared the notes and comment and most of the exercises.

Throughout the work, we have had the good fortune of enjoying Jaak Peetre's kind patronage and invaluable counsel. We want to express our deep gratitude to him. Thanks are also due to our colleagues for their support and help. Finally, we are sincerely grateful to Boel Engebrand, Lena Mattsson and Birgit Höglund for their expert typing of our manuscript.

This is the first attempt, as far as we know, to treat interpolation theory fairly comprehensively in book form. Perhaps this fact could partly excuse the many shortcomings, omissions and inconsistencies of which we may be guilty. We beg for all information about such insufficiencies and for any constructive criticism.

Lund and Göteborg, January 1976

Jöran Bergh Jörgen Löfström

Introduction

In recent years, there has emerged a new field of study in functional analysis: the theory of interpolation spaces. Interpolation theory has been applied to other branches of analysis (e.g. partial differential equations, numerical analysis, approximation theory), but it has also attracted considerable interest in itself. We intend to give an introduction to the theory, thereby covering the main elementary results.

In Chapter 1, we present the classical interpolation theorems of Riesz-Thorin and Marcinkiewicz with direct proofs, and also a few applications. The notation and the basic concepts are introduced in Chapter 2, where we also discuss some general results, e.g. the Aronszajn-Gagliardo theorem.

We treat two essentially different interpolation methods: the real method and the complex method. These two methods are modelled on the proofs of the Marcinkiewicz theorem and the Riesz-Thorin theorem respectively, as they are given in Chapter 1. The real method is presented, following Peetre, in Chapter 3; the complex method, following Calderón, in Chapter 4.

Chapter 5—7 contain applications of the general methods expounded in Chapter 3 and 4.

In Chapter 5, we consider interpolation of L_p-spaces, including general versions of the interpolation theorems of Riesz-Thorin, and of Marcinkiewicz, as well as other results, for instance, the theorem of Stein-Weiss concerning the interpolation of L_p-spaces with weights.

Chapter 6 contains the interpolation of Besov spaces and generalized Sobolev spaces (defined by means of Bessel potentials). We use the definition of the Besov spaces given by Peetre. We list the most important interpolation results for these spaces, and present various inclusion theorems, a general version of Sobolev's embedding theorem and a trace theorem. We also touch upon the theory of semi-groups of operators.

In Chapter 7 we discuss the close relation between interpolation theory and approximation theory (in a wide sense). We give some applications to classical approximation theory and theoretical numerical analysis.

We have emphasized the real method at the expense of a balance (with respect to applicability) between the real and the complex method. A reason for this is that the real interpolation theory, in contrast to the case of the complex theory, has not been treated comprehensively in one work. As a consequence, whenever it is possible to use both the real and the complex method, we have preferred to apply the real method.

In each chapter the penultimate section contains exercises. These are meant to extend and complement the results of the previous sections. Occasionally, we use the content of an exercise in the subsequent main text. We have tried to give references for the exercises. Moreover, many important results and most of the applications can be found only as exercises.

Concluding each chapter, we have a section with notes and comment. These include historical sketches, various generalizations, related questions and references. However, we have not aimed at completeness: the historical references are not necessarily the first ones; many papers worth mention have been left out. By giving a few key references, i.e. those which are pertinent to the reader's own further study, we hope to compensate partly for this.

The potential reader we have had in mind is conversant with the elements of real (several variables) and complex (one variable) analysis, of Fourier analysis, and of functional analysis. Beyond an elementary level, we have tried to supply proofs of the statements in the main text. Our general reference for elementary results is Dunford-Schwartz [1].

We use some symbols with a special convention or meaning. For other notation, see the Index of Symbols.

$f(x) \sim g(x)$ "There are positive constants C_1 and C_2 such that $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$ (f and g being non-negative functions)."
Read: f and g are equivalent.

$T: A \to B$ "T is a continuous mapping from A to B."

$A \subset B$ "A is continuously embedded in B."

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Chapter 1

Some Classical Theorems

The classical results which provided the main impetus for the study of interpolation *in se* are the theorems of M. Riesz, with Thorin's proof, and of Marcinkiewicz. Thorin's proof of the Riesz-Thorin theorem contains the idea behind the complex interpolation method. Analogously, the way of proving the Marcinkiewicz theorem resembles the construction of the real interpolation method. We give direct proofs of these theorems (Section 1.1 and Section 1.3), and a few of their applications (Section 1.2 and Section 1.4). More recently, interpolation methods have been used in approximation theory. In Section 1.5 we rewrite the classical Bernstein and Jackson inequalities to indicate the connection with approximation theory.

The purpose of this chapter is to introduce the type of theorems which will be proved later, and also to give a first hint of the techniques used in their proofs. Note that, in this introductory chapter, we are not stating the results in the more general form they will have in later chapters.

1.1. The Riesz-Thorin Theorem

Let (U, μ) be a measure space, μ always being a positive measure. We adopt the usual convention that two functions are considered equal if they agree except on a set of μ-measure zero. Then we denote by $L_p(U, d\mu)$ (or simply $L_p(d\mu)$, $L_p(U)$ or even L_p) the Lebesgue-space of (all equivalence classes of) scalar-valued μ-measurable functions f on U, such that

(1) $$\|f\|_{L_p} = (\textstyle\int_U |f(x)|^p d\mu)^{1/p}$$

is finite. Here we have $1 \leqslant p < \infty$. In the limiting case, $p = \infty$, L_p consists of all μ-measurable and bounded functions. Then we write

(2) $$\|f\|_{L_\infty} = \sup_U |f(x)| .$$

In this section and the next, scalars are supposed to be complex numbers.

Let T be a linear mapping from $L_p = L_p(U, d\mu)$ to $L_q = L_q(V, d\nu)$. This means that $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$. We shall write

$$T: L_p \to L_q$$

if in addition T is bounded, i.e. if

$$M = \sup_{f \neq 0} \| Tf \|_{L_q} / \| f \|_{L_p}$$

is finite. The number M is called the norm of the mapping T.

Now we have the following well-known theorem.

1.1.1. Theorem (The Riesz-Thorin interpolation theorem). *Assume that* $p_0 \neq p_1$, $q_0 \neq q_1$ *and that*

$$T: L_{p_0}(U, d\mu) \to L_{q_0}(V, d\nu)$$

with norm M_0, *and that*

$$T: L_{p_1}(U, d\mu) \to L_{q_1}(V, d\nu)$$

with norm M_1. *Then*

$$T: L_p(U, d\mu) \to L_q(V, d\nu)$$

with norm

(3) $$M \leqslant M_0^{1-\theta} M_1^{\theta}$$

provided that $0 < \theta < 1$ *and*

(4) $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Note that (3) means that M is logarithmically convex, i.e. $\log M$ is convex. Note also the geometrical meaning of (4). The points $(1/p, 1/q)$ described by (4)

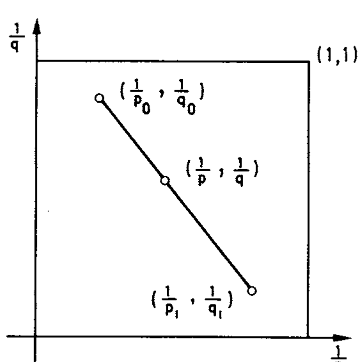


Fig. 1

are the points on the line segment between $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$. (Obviously one should think of L_p as a "function" of $1/p$ rather than of p.)

Later on we shall prove the Riesz-Thorin interpolation (or convexity) theorem by means of abstract methods. Here we shall reproduce the elementary proof which was given by Thorin.

Proof: Let us write

$$\langle h, g\rangle = \int_V h(y)g(y)dv$$

and $1/q' = 1 - 1/q$. Then we have, by Hölder's inequality,

$$\|h\|_{L_q} = \sup\{|\langle h, g\rangle| : \|g\|_{L_{q'}} = 1\}.$$

and

$$M = \sup\{|\langle Tf, g\rangle| : \|f\|_{L_p} = \|g\|_{L_{q'}} = 1\}.$$

Since $p < \infty, q' < \infty$ we can assume that f and g are bounded with compact supports.

For $0 \leqslant \operatorname{Re} z \leqslant 1$ we put

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \qquad \frac{1}{q'(z)} = \frac{1-z}{q'_0} + \frac{z}{q'_1},$$

and

$$\varphi(z) = \varphi(x, z) = |f(x)|^{p/p(z)} f(x)/|f(x)|, \qquad x \in U,$$
$$\psi(z) = \psi(y, z) = |g(y)|^{q'/q'(z)} g(y)/|g(y)|, \qquad y \in V.$$

It follows that $\varphi(z) \in L_{p_j}$ and $\psi(z) \in L_{q'_j}$ and hence that $T\varphi(z) \in L_{q_j}$, $j = 0, 1$. It is also easy to see that $\varphi'(z) \in L_{p_j}$, $\psi'(z) \in L_{q_j}$ and thus also that $(T\varphi)'(z) \in L_{q_j}$, $(0 < \operatorname{Re} z < 1)$. This implies the existence of

$$F(z) = \langle T\varphi(z), \psi(z)\rangle, \qquad 0 \leqslant \operatorname{Re} z \leqslant 1.$$

Moreover it follows that $F(z)$ is analytic on the open strip $0 < \operatorname{Re} z < 1$, and bounded and continuous on the closed strip $0 \leqslant \operatorname{Re} z \leqslant 1$.

Next we note that

$$\|\varphi(it)\|_{L_{p_0}} = \|\,|f|^{p/p_0}\|_{L_{p_0}} = \|f\|_{L_p}^{p/p_0} = 1,$$
$$\|\varphi(1+it)\|_{L_{p_1}} = \|\,|f|^{p/p_1}\|_{L_{p_1}} = \|f\|_{L_p}^{p/p_1} = 1,$$

and similarly

$$\|\psi(it)\|_{L_{q'_0}} = \|\psi(1+it)\|_{L_{q'_1}} = 1.$$

By the assumptions, we therefore have

$$(3)\qquad \begin{aligned} |F(it)| &\leqslant \|T\varphi(it)\|_{L_{q_0}} \cdot \|\psi(it)\|_{L_{q_0'}} \leqslant M_0, \\ |F(1+it)| &\leqslant \|T\varphi(1+it)\|_{L_{q_1}} \cdot \|\psi(1+it)\|_{L_{q_1'}} \leqslant M_1. \end{aligned}$$

We also note that

$$\varphi(\theta)=f, \qquad \psi(\theta)=g,$$

and thus

$$(4)\qquad F(\theta)=\langle Tf, g\rangle.$$

Using now the three line theorem (a variant of the well-known Hadamard three circle theorem), reproduced as Lemma 1.1.2 below, we get the conclusion

$$|\langle Tf, g\rangle| \leqslant M_0^{1-\theta} M_1^{\theta},$$

or equivalently

$$M \leqslant M_0^{1-\theta} M_1^{\theta}. \quad \square$$

1.1.2. Lemma (The three line theorem). *Assume that $F(z)$ is analytic on the open strip $0<\operatorname{Re} z<1$ and bounded and continuous on the closed strip $0\leqslant \operatorname{Re} z \leqslant 1$. If*

$$|F(it)| \leqslant M_0, \qquad |F(1+it)| \leqslant M_1, \qquad -\infty<t<\infty,$$

we then have

$$|F(\theta+it)| \leqslant M_0^{1-\theta} M_1^{\theta}, \qquad -\infty<t<\infty.$$

Proof: Let ε be a positive and λ an arbitrary real number. Put

$$F_\varepsilon(z)=\exp(\varepsilon z^2+\lambda z)\, F(z).$$

Then it follows that

$$F_\varepsilon(z) \to 0 \quad \text{as} \quad \operatorname{Im} z \to \pm\infty,$$

and

$$|F_\varepsilon(it)| \leqslant M_0, \qquad |F_\varepsilon(1+it)| \leqslant M_1 e^{\varepsilon+\lambda}.$$

By the Phragmén-Lindelöf principle we therefore obtain

$$|F_\varepsilon(z)| \leqslant \max(M_0, M_1 e^{\varepsilon+\lambda}),$$

i.e.,

$$|F(\theta+it)| \leqslant \exp(-\varepsilon(\theta^2-t^2)) \max(M_0 e^{-\theta\lambda}, M_1 e^{(1-\theta)\lambda+\varepsilon}).$$

This holds for any fixed θ and t. Letting $\varepsilon \to 0$ we conclude that

$$|F(\theta+it)| \leqslant \max(M_0\rho^{-\theta}, M_1\rho^{1-\theta})$$

where $\rho = \exp\lambda$. The right hand side is as small as possible when $M_0\rho^{-\theta} = M_1\rho^{1-\theta}$, i.e. when $\rho = M_0/M_1$. With this choice of ρ we get

$$|F(\theta+it)| \leqslant M_0^{1-\theta} M_1^{\theta}. \quad \square$$

1.2. Applications of the Riesz-Thorin Theorem

In this section we shall give two rather simple applications of the Riesz-Thorin interpolation theorem. We include them here in order to illustrate the rôle of interpolation theorems of which the Riesz-Thorin theorem is just one (albeit important) example.

We shall consider the case $U = V = \mathbb{R}^n$ and $d\mu = dv = dx$ (Lebesgue-measure). We let T be the Fourier transform $\mathscr{F}$ defined by

$$(\mathscr{F}f)(\xi) = \hat{f}(\xi) = \int f(x) \exp(-i\langle x, \xi\rangle) dx,$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$. Here $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$. Then we have

$$|\mathscr{F}f(\xi)| \leqslant \int |f(x)| dx$$

and by Parseval's formula

$$\int |\mathscr{F}f(\xi)|^2 d\xi = (2\pi)^n \int |f(x)|^2 dx.$$

This means that

$$\mathscr{F}: L_1 \to L_\infty, \quad \text{norm } 1,$$

$$\mathscr{F}: L_2 \to L_2, \quad \text{norm } (2\pi)^{n/2}.$$

Using the Riesz-Thorin theorem, we conclude that

(1) $$\mathscr{F}: L_p \to L_q$$

with

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}, \quad 0 < \theta < 1.$$

Eliminating θ, we see that $1/p = 1 - 1/q$, i.e., $q = p'$, where $1 < p < 2$. The norm of the mapping (1) is bounded by $(2\pi)^{n\theta/2} = (2\pi)^{n/p'}$. We have proved the following result.

1.2.1. Theorem (The Hausdorff-Young inequality). *If* $1 \leqslant p \leqslant 2$ *we have*

$$\|\mathscr{F}f\|_{L_{p'}} \leqslant (2\pi)^{n/p'} \|f\|_{L_p}. \quad \square$$

As a second application of the Riesz-Thorin theorem we consider the convolution operator

$$Tf(x) = \int k(x-y)f(y)dy = k * f(x)$$

where k is a given function in L_ρ. By Minkowski's inequality we have

$$\|Tf\|_{L_\rho} \leqslant \|k\|_{L_\rho} \|f\|_{L_1},$$

and, by Hölder's inequality,

$$\|Tf\|_{L_\infty} \leqslant \|k\|_{L_\rho} \|f\|_{L_{\rho'}}.$$

Thus

$$T: L_1 \to L_\rho,$$

$$T: L_{\rho'} \to L_\infty,$$

and therefore

$$T: L_p \to L_q$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\rho'}, \quad \frac{1}{q} = \frac{1-\theta}{\rho} + \frac{\theta}{\infty}.$$

Elimination of θ yields $1/q = 1/p - 1/\rho'$ and $1 \leqslant p \leqslant \rho'$. This gives the following result.

1.2.2. Theorem (Young's inequality). *If* $k \in L_\rho$ *and* $f \in L_p$ *where* $1 < p < \rho'$ *then* $k * f \in L_q$ *for* $1/q = 1/p - 1/\rho'$ *and*

$$\|k * f\|_{L_q} \leqslant \|k\|_{L_\rho} \|f\|_{L_p}. \quad \square$$

1.3. The Marcinkiewicz Theorem

Consider again the measure space (U, μ). In this section the scalars may be real or complex. If f is a scalar-valued μ-measurable function which is finite almost everywhere, we introduce the distribution function $m(\sigma, f)$ defined by

$$m(\sigma, f) = \mu(\{x : |f(x)| > \sigma\}).$$

Since we have assumed that μ is positive, we have that $m(\sigma, f)$ is a real-valued or extended real-valued function of σ, defined on the positive real axis $\mathbb{R}_+ = (0, \infty)$. Clearly $m(\sigma, f)$ is non-increasing and continuous on the right. Moreover, we have

$$(1) \qquad \|f\|_{L_p} = (p\int_0^\infty \sigma^p m(\sigma, f)\, d\sigma/\sigma)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty\,,$$

and

$$(2) \qquad \|f\|_{L_\infty} = \inf\{\sigma : m(\sigma, f) = 0\}\,.$$

Using the distribution function $m(\sigma, f)$, we now introduce the weak L_p-spaces denoted by L_p^*. The space L_p^*, $1 \leqslant p < \infty$, consists of all f such that

$$\|f\|_{L_p^*} = \sup_\sigma \sigma m(\sigma, f)^{1/p} < \infty\,.$$

In the limiting case $p = \infty$ we put $L_\infty^* = L_\infty$. Note that $\|f\|_{L_p^*}$ is not a norm if $1 \leqslant p < \infty$. In fact, it is clear that

$$(3) \qquad m(\sigma, f+g) \leqslant m(\sigma/2, f) + m(\sigma/2, g)\,.$$

Using the inequality $(a+b)^{1/p} \leqslant a^{1/p} + b^{1/p}$, we conclude that

$$\|f+g\|_{L_p^*} \leqslant 2(\|f\|_{L_p^*} + \|g\|_{L_p^*})\,.$$

This means that L_p^* is a so called quasi-normed vector space. (In a normed space we have the triangle inequality $\|f+g\| \leqslant \|f\| + \|g\|$, but in a quasi-normed space we only have the quasi-triangle inequality $\|f+g\| \leqslant k(\|f\| + \|g\|)$ for some $k \geqslant 1$.) If $p > 1$ one can, however, as will be seen later on, find a norm on L_p^* and, with this norm, L_p^* becomes a Banach space. One can show that L_1^* is complete but not a normable space. (See Section 1.6.)

The spaces L_p^* are special cases of the more general Lorentz spaces L_{pr}. In their definition we use yet another concept. If f is a μ-measurable function we denote by f^* its *decreasing rearrangement*

$$(4) \qquad f^*(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}\,.$$

This is a non-negative and non-increasing function on $(0, \infty)$ which is continuous on the right and has the property

$$(5) \qquad m(\rho, f^*) = m(\rho, f)\,, \qquad \rho \geqslant 0\,.$$

(See Figure 2.) Thus f^* is equimeasurable with f. In fact, by (4) we have $f^*(m(\rho, f)) \leqslant \rho$ and thus $m(\rho, f^*) \leqslant m(\rho, f)$. Moreover, since f^* is continuous on the right, $f^*(m(\rho, f^*)) \leqslant \rho$ and hence $m(\rho, f) \leqslant m(\rho, f^*)$.

Note that at all points t where $f^*(t)$ is continuous the relation $\sigma = f^*(t)$ is equivalent to $t = m(\sigma, f)$.

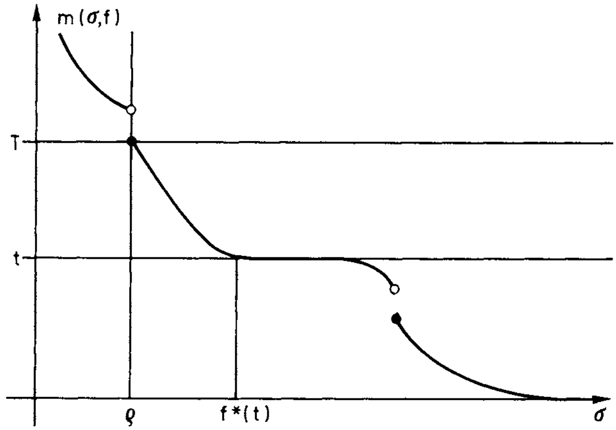


Fig. 2

Now the Lorentz space L_{pr} is defined as follows. We have $f \in L_{pr}$, $1 \leqslant p \leqslant \infty$, if and only if

$$\|f\|_{L_{pr}} = (\int_0^\infty (t^{1/p} f^*(t))^r dt/t)^{1/r} < \infty \quad \text{when} \quad 1 \leqslant r < \infty,$$

$$\|f\|_{L_{p\infty}} = \sup_t t^{1/p} f^*(t) < \infty \quad \text{when} \quad r = \infty.$$

We have the following, with equality of norms,

$$\begin{aligned} L_{pp} &= L_p \\ L_{p\infty} &= L_p^* \end{aligned} \qquad 1 \leqslant p \leqslant \infty.$$

These statements are implied by (1), (2), (4) and (5); only the last one is not immediate when $1 \leqslant p < \infty$. If for a given σ there is a t such that $f^*(t) = \sigma$ then by (4) we have $m(\sigma, f) \leqslant t$. Thus $\sigma m(f, \sigma)^{1/p} \leqslant t^{1/p} f^*(t)$ which implies $\|f\|_{L_p^*} \leqslant \|f\|_{L_{p\infty}}$. On the other hand, given $\varepsilon > 0$, we can choose t as a point of continuity of $f^*(t)$ such that $\|f\|_{L_{p\infty}} \leqslant t^{1/p} f^*(t) + \varepsilon$. Put $\sigma = f^*(t)$. Then $m(\sigma, f) = t$ and $\|f\|_{L_{p\infty}} \leqslant t^{1/p} f^*(t) + \varepsilon = \sigma m(\sigma, f)^{1/p} + \varepsilon \leqslant \|f\|_{L_p^*} + \varepsilon$, which completes the proof. □

In general L_{pr} is a quasi-normed space, but when $p > 1$ it is possible to replace the quasi-norm with a norm, which makes L_{pr} a Banach space. (See Section 1.6.)

It is possible to prove that $L_{pr_1} \subset L_{pr_2}$ if $r_1 \leqslant r_2$. (See Section 1.6.) Taking $r_1 = p$ and $r_2 = \infty$ we obtain, in particular,

(6) $\qquad L_p \subset L_p^*.$

This also follows directly from the definition (3) of L_p^*. In fact,

$$\|f\|_{L_p} = \sup_\sigma (\int_{|f(x)| > \sigma} |f(x)|^p d\mu)^{1/p} \geqslant \sup_\sigma \sigma m(\sigma, f)^{1/p} = \|f\|_{L_p^*}.$$

We shall consider linear mappings T from L_p to L_q^*. Such a mapping is said to be bounded if $\|Tf\|_{L_q^*} \leqslant C\|f\|_{L_p}$, and the infimum over all possible numbers C is called the norm of T. We then write $T: L_p \to L_q^*$. We are ready to state and prove the following important interpolation theorem.

1.3.1. Theorem (The Marcinkiewicz interpolation theorem). *Assume that* $p_0 \neq p_1$ *and that*

$$T: L_{p_0}(U, d\mu) \to L_{q_0}^*(V, dv) \quad \textit{with norm } M_0^*,$$

$$T: L_{p_1}(U, d\mu) \to L_{q_1}^*(V, dv) \quad \textit{with norm } M_1^*.$$

Put

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

and assume that

$$p \leqslant q. \tag{7}$$

Then

$$T: L_p(U, d\mu) \to L_q(V, dv)$$

with norm M *satisfying*

$$M \leqslant C_\theta M_0^{*1-\theta} M_1^{*\theta}.$$

This theorem, although certainly reminiscent of the Riesz-Thorin theorem, differs from it in several important respects. Among other things, we note that scalars may be real or complex numbers, but in the Riesz-Thorin theorem we must insist on complex scalars. (Otherwise we can only prove the convexity inequality $M \leqslant CM_0^{1-\theta} M_1^\theta$.) On the other hand, there is the restriction (7). The most important feature is, however, that we have replaced the spaces L_{q_0} and L_{q_1} by the larger spaces $L_{q_0}^*$ and $L_{q_1}^*$ in the assumption. Therefore the Marcinkiewicz theorem can be used in cases where the Riesz-Thorin theorem fails.

Proof: We shall give a complete proof of this theorem in Chapter 5 (see Theorem 5.3.2). Here we shall consider only the case $p_0 = q_0$, $p_1 = q_1$, and $1 \leqslant p_0 < p_1 < \infty$, and non-atomic measure on, say, $\mathbb{R}^n$.

Moreover, we shall prove only the estimate

$$M \leqslant C \max(M_0^*, M_1^*).$$

In order to prove this, it will clearly be sufficient to assume that $M_0^* \leqslant 1$ and $M_1^* \leqslant 1$.

Put

$$f_0(x) = f_0(t, x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_1(x)=f_1(t,x)=f(x)-f_0(t,x),$$

where $E\subset\{x:|f(x)|\geqslant f^*(t)\}$ is chosen with $\mu(E)=\min(t,\mu(U))$.
Using (3) and the linearity of T, we see that

$$(Tf)^*(t)\leqslant(Tf_0)^*(t/2)+(Tf_1)^*(t/2).$$

By the assumptions on T we have

$$(Tf_i)^*(t/2)\leqslant Ct^{-1/p_i}\|f_i\|_{L_{p_i}},\qquad i=0,1.$$

It follows that

$$\|Tf\|_{L_p}=(\int_0^\infty(t^{-1}(Tf)^*(t))^p\,dt)^{1/p}\leqslant C(I_0+I_1),$$

where

$$I_0=(\int_0^\infty(t^{-1}\|f_0\|_{L_{p_0}}^{p_0})^{p/p_0}dt)^{1/p}=(\int_0^\infty(t^{-1}\int_0^t(f^*(s))^{p_0}ds)^{p/p_0}dt)^{1/p},$$

and

$$I_1=(\int_0^\infty(t^{-1}\|f_1\|_{L_{p_1}}^{p_1})^{p/p_1}dt)^{1/p}=(\int_0^\infty(t^{-1}\int_t^\infty(f^*(s))^{p_1}ds)^{p/p_1}dt)^{1/p}.$$

In order to estimate I_0 we use Minkowski's inequality to obtain

$$I_0^p=\int_0^\infty(\int_0^1(f^*(t\sigma))^{p_0}d\sigma)^{p/p_0}dt\leqslant(\int_0^1(\int_0^\infty(f(t\sigma))^p dt)^{p_0/p}d\sigma)^p\leqslant C\|f\|_{L_p}^p.$$

In order to estimate I_1, we use the inequality

(8) $$\int_0^\infty(t^{-1}\int_t^\infty\varphi^*(s)ds)^\theta dt\leqslant C\int_0^\infty(\varphi^*(s))^\theta ds,\qquad 0<\theta<1.$$

Using this estimate with $\theta=p/p_1$ and $\varphi=|f|^{p_1}$, we obtain, noting also that $\varphi^*=(f^*)^{p_1}$,

$$I_1^p\leqslant C\|f\|_{L_p}^p.$$

Thus

$$\|Tf\|_{L_p}\leqslant C\|f\|_{L_p},$$

which concludes the proof. It remains, however, to prove (8).

In order to prove (8) we put $a_\mu=\varphi^*(2^\mu)$. Since $\varphi^*(t)$ and $t^{-1}\int_t^\infty\varphi^*(s)ds$ are decreasing functions of t, we have

$$\int_0^\infty(t^{-1}\int_t^\infty\varphi^*(s)ds)^\theta dt\leqslant C\sum_{\nu=-\infty}^{\infty}(2^{-\nu}\sum_{\mu\geqslant\nu}a_\mu 2^\mu)^\theta 2^\nu.$$

Since $(x+y)^\theta\leqslant x^\theta+y^\theta$ for $0<\theta<1$, we can estimate the right hand side by a constant multiplied by

$$\sum_\nu\sum_{\mu\geqslant\nu}2^{(1-\theta)\nu}a_\mu^\theta 2^{\theta\mu}=\sum_\mu a_\mu^\theta 2^{\mu\theta}\sum_{\nu\geqslant\mu}2^{\nu(1-\theta)}.$$

It follows that

$$\int_0^\infty (t^{-1}\int_t^\infty \varphi^*(s)\,ds)^\theta dt \leqslant C\sum_\mu a_\mu^\theta 2^\mu \leqslant C\int_0^\infty (\varphi^*(s))^\theta ds . \quad \square$$

1.4. An Application of the Marcinkiewicz Theorem

We shall prove a generalization of the Hausdorff-Young inequality due to Payley. We consider the measure space $(\mathbb{R}^n, \mu)$, μ Lebesgue measure. Let w be a weight function on $\mathbb{R}^n$, i.e. a positive and measurable function on $\mathbb{R}^n$. Then we denote by $L_p(w)$ the L_p-space with respect to $w\,dx$. The norm on $L_p(w)$ is

$$\|f\|_{L_p(w)} = (\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx)^{1/p} .$$

With this notation we have the following theorem.

1.4.1. Theorem. *Assume that* $1 \leqslant p \leqslant 2$. *Then*

(1) $$\|\mathscr{F} f\|_{L_p(|\xi|^{-n(2-p)})} \leqslant C_p \|f\|_{L_p} .$$

Proof: We consider the mapping

$$(Tf)(\xi) = |\xi|^n \hat{f}(\xi) .$$

By Parseval's formula, we have

$$\|Tf\|_{L_2(|\xi|^{-2n})} = \|\hat{f}\|_{L_2} \leqslant C\|f\|_{L_2} .$$

Since $L_2^*(|\xi|^{-2n}) \supset L_2(|\xi|^{-2n})$, we conclude that

(2) $$T: L_2 \to L_2^*(|\xi|^{-2n}) .$$

We now claim that

(3) $$T: L_1 \to L_1^*(|\xi|^{-2n}) .$$

Applying the Marcinkiewicz interpolation theorem we obtain

$$T: L_p \to L_p(|\xi|^{-2n})$$

which implies the theorem.

In order to prove (3) we consider the set

$$E_\sigma = \{\xi : |\xi|^n |\hat{f}(\xi)| > \sigma\} .$$

Let us write ν for the measure $|\xi|^{-2n}d\xi$ and assume that $\|f\|_{L_1}=1$. Then $|\hat{f}(\xi)|\leqslant 1$. For $\xi\in E_\sigma$ we therefore have $\sigma\leqslant|\xi|^n$. Consequently

$$m(\sigma, Tf)=\nu(E_\sigma)=\int_{E_\sigma}|\xi|^{-2n}d\xi\leqslant\int_{|\xi|^n\geqslant\sigma}|\xi|^{-2n}d\xi\leqslant C\sigma^{-1}.$$

This proves that

$$\sigma m(\sigma, Tf)\leqslant C\|f\|_{L_1},$$

i.e. (3) holds. □

1.5. Two Classical Approximation Results

A characteristic feature of interpolation theory is the convexity inequality $M\leqslant M_0^{1-\theta}M_1^\theta$. When an inequality of this form appears there is often a connection with interpolation theory. In this section we rewrite the classical Bernstein inequality as a convexity inequality, thereby indicating a connection between classical approximation theory and interpolation theory. Also, the converse inequality, the Jackson inequality, is reformulated as an inequality which is "dual" to the convexity inequality above. These topics will be discussed in greater detail in Chapter 7.

Let $\mathbb{T}$ be the one-dimensional torus. Then we may write Bernstein's inequality as follows:

$$\sup_{\mathbb{T}}|D^j a(x)|\leqslant n^j\sup_{\mathbb{T}}|a(x)|, \quad j=0,1,2,\dots, \tag{1}$$

where a is a trigonometric polynomial of degree at most n. In order to reformulate (1), put

$A_0=\{\text{trigonometric polynomials}\}$,
$A_1=\{\text{continuous } 2\pi\text{-periodic functions}\}$,
$A_\theta=\{2\pi\text{-periodic functions } a \text{ with } D^j a\in A_1\}$, $\theta=1/(j+1)$,

$\|a\|_{A_0}=(\text{the degree of } a)^{1/(j+1)}$,
$\|a\|_{A_1}=\sup_{\mathbb{T}}|a(x)|^{1/(j+1)}$,
$\|a\|_{A_\theta}=\sup_{\mathbb{T}}|D^j a(x)|^{1/(j+1)}$.

Note that the last three expressions are not norms. In addition, scalar multiplication is not continuous in $\|\cdot\|_{A_0}$. With this notation, (1) may be rewritten as

$$\|a\|_{A_\theta}\leqslant\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^\theta, \quad 0<\theta\leqslant 1, \quad a\in A_0\cap A_1. \tag{1'}$$

Clearly, (1′) resembles, at least formally, the convexity inequalities in the theorems of Riesz-Thorin and Marcinkiewicz. The other classical inequality is of Jackson type:

$$\inf \sup_{\mathbb{T}}|a(x)-a_0(x)|\leqslant Cn^{-j}\sup_{\mathbb{T}}|D^j a(x)|, \tag{2}$$

where "inf" is taken over all trigonometric polynomicals a_0 of degree at most n, and a is a j-times continuously differentiable 2π-periodic function. Using the notation introduced above and writing $a_1 = a - a_0$, we have the following version of (2):

for each $a \in A_\theta$ *and for each* n *there exist* $a_0 \in A_0$ *and* $a_1 \in A_1$, *with* $a_0 + a_1 = a$ $(\in A_0 + A_1)$, *such that*

$$(2') \qquad \begin{array}{l} \|a_0\|_{A_0} \leqslant C n^\theta \|a\|_{A_\theta} \\ \|a_1\|_{A_1} \leqslant C n^{\theta-1} \|a\|_{A_\theta} \end{array} \qquad (0 < \theta \leqslant 1)$$

Evidently, (2′) is, in a sense, dual to (1′).

1.6. Exercises

1. (a) (Schur [1]). Let $l_p = \{x = (x_i)_{i=1}^\infty;\ x_i \in \mathbb{C},\ (\sum_{i=1}^\infty |x_i|^p)^{1/p} < \infty\}$, $1 \leqslant p \leqslant \infty$, with the norm $\|x\|_{l_p} = (\sum_{i=1}^\infty |x_i|^p)^{1/p}$, $\|x\|_{l_\infty} = \max_i |x_i|$. Let $A = (a_{ij})_{i,j=1}^\infty$, $a_{ij} \in \mathbb{C}$, be a matrix. Show (without using Riesz-Thorin) that

$$\begin{cases} A: l_1 \to l_1 \\ A: l_\infty \to l_\infty \end{cases} \Rightarrow A: l_2 \to l_2,$$

and that

$$\|A\|_{l_2} \leqslant \|A\|_{l_1}^{1/2} \|A\|_{l_\infty}^{1/2}$$

holds for the norms of A.

Hint: Write $|a_{ij}| = |a_{ij}|^{1/2} \cdot |a_{ij}|^{1/2}$ and use the Cauchy-Schwarz inequality.

(b) Show that if

$$n|a_{nj}| \leqslant M_0,$$
$$\textstyle\sum_j |a_{nj}| \leqslant M_1$$

then $A: l_p \to l_p$ $(1 < p \leqslant \infty)$ and

$$\|A\|_{l_p} \leqslant C M_0^{1/p} M_1^{1/p'}.$$

Hint: Prove that $A: l_1 \to l_1^*$.

(c) Show that the conclusion in (a) may be strengthened to $A: l_p \to l_p$ and

$$\|A\|_{l_p} \leqslant \|A\|_{l_1}^{1/p} \|A\|_{l_\infty}^{1/p'} \qquad (1 \leqslant p \leqslant \infty).$$

2. (F. Riesz [1]). In the notation of the previous exercise, show that if A is unitary, i.e.

$$\sum_{j=1}^{\infty} a_{ij}\bar{a}_{kj} = \delta_{ik} \quad \text{(Kronecker delta)},$$

and $\sup_{i,j}|a_{ij}|$ is finite then

$$A: l_p \to l_{p'}, \quad 1 \leqslant p \leqslant 2, \quad 1/p + 1/p' = 1.$$

Moreover, prove that for the norms

$$\|A\|_{l_p \to l_{p'}} \leqslant \sup_{i,j}|a_{ij}|^{1/p - 1/p'}.$$

(Cf. the Hausdorff-Young theorem.)
Hint: Show that $A: l_1 \to l_\infty$ and $A: l_2 \to l_2$.

3. Let $f \in L_p(\mathbb{T})$, $\mathbb{T}$ being the one dimensional torus, $1 \leqslant p \leqslant \infty$, and assume that f has the Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

For a given sequence $\lambda = (\lambda_n)_{n=-\infty}^{\infty}$, let λf be defined by the Fourier series

$$\sum_{n=-\infty}^{\infty} \lambda_n c_n e^{inx}.$$

Put

$$m_p = \{\lambda: \|\lambda f\|_p \leqslant C\|f\|_p \text{ for all } f \in L_p(T)\},$$

and let $\|\lambda\|_{m_p}$ denote the norm of the mapping $f \to \lambda f$. Show that, with $1/p + 1/p' = 1$,

(i) $m_p = m_{p'}$, $1 \leqslant p \leqslant \infty$;
(ii) $\lambda \in m_1 \Leftrightarrow \sum_n |\lambda_n| < \infty$;
(iii) $\lambda \in m_2 \Leftrightarrow \sup_n |\lambda_n| < \infty$;
(iv) if $\lambda \in m_{p_0} \cap m_{p_1}$, $1 \leqslant p_0$, $p_1 \leqslant \infty$ then $\lambda \in m_p$, and

$$\|\lambda\|_{m_p} \leqslant \|\lambda\|_{m_{p_0}}^{1-\theta} \|\lambda\|_{m_{p_1}}^{\theta}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1.$$

Hint: Apply Hölder's inequality to the integral $\int_0^{2\pi} \lambda f(x) g(x)\,dx$.

4. (M. Riesz [2]). Prove that if $1 < p < \infty$ and $f \in L_p$, with

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{a=1}^{\infty} a_n \cos nx + b_n \sin nx \quad (a_n, b_n \in \mathbb{R}),$$

then the conjugate function $\tilde{f} \in L_p$, with

$$\tilde{f}(x) \sim \sum_{n=1}^{\infty} a_n \sin nx - b_n \cos nx\,,$$

and

$$\|\tilde{f}\|_{L_p} \leqslant C_p \|f\|_{L_p}\,.$$

Show that this result is equivalent to $\lambda \in m_p$, $1 < p < \infty$, where

$$\lambda_n = \begin{cases} 1, & n > 0 \\ 0, & n \leqslant 0\,. \end{cases}$$

Hint: (i) Apply Cauchy's integral theorem to show that, for $p = 2, 4, 6, \ldots$, $\int_{-\pi}^{\pi} (f(x) + i\tilde{f}(x))^p \, dx = \pi a_0$. Consider the real part.
(ii) Note that $\lambda f = f + i\tilde{f}$, in some sense.
(iii) Use Exercise 3 to get the whole result.

5. (a) (M. Riesz [2]). With $f \in L_p(\mathbb{T})$ $(1 < p < \infty)$ as in Exercise 3, define the Hilbert transform of f on $\mathbb{T}$ by

$$\mathscr{H} f(x) = \int_{\mathbb{T}} \frac{f(y)}{1 - \exp(i(x-y))}\, dy\,,$$

where the integral denotes the Cauchy principal value. Show that, with λ as in Exercise 4, $\lambda - \frac{1}{2} = \mathscr{H}$. Use this to establish that

$$\|\mathscr{H} f\|_{L_p} \leqslant C_p \|f\|_{L_p}\,, \qquad 1 < p < \infty\,.$$

Hint: Apply the residue theorem of complex function theory.

(b) (O'Neil-Weiss [1]). Consider now the real line $\mathbb{R}$ and define the Hilbert transform $\mathscr{H}$ by

$$\mathscr{H} f(x) = \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy\,,$$

where the integral is the Cauchy principal value. Let E be a Lebesgue-measurable subset of $\mathbb{R}$ with finite measure $|E|$. Then

$$(\mathscr{H}\chi_E)^*(t) = \pi^{-1} \sinh^{-1}(2|E|/t)\,.$$

Prove that this implies that if the integral

$$\int_0^{\infty} f^*(t) \sinh^{-1}(1/t)\, dt$$

is finite then

$$\int_0^s (\mathscr{H} f)^*(s)\, ds \leqslant 2\pi^{-1} \int_0^{\infty} f^*(t) \sinh^{-1}(s/t)\, dt \qquad (s > 0)\,.$$

Use this inequality to obtain

$$\mathscr{H}: L_p \to L_p \quad (1<p<\infty).$$

6. Show that L_{pr} defined in Section 1.3 is complete if $1<p<\infty$ and $1\leqslant r\leqslant\infty$ or if $p=r=1,\infty$; and that $\|f+g\|_{L_{pr}}\leqslant\|f\|_{L_{pr}}+\|g\|_{L_{pr}}$ iff $1\leqslant r\leqslant p<\infty$ or $p=r=\infty$. For which p and r is L_{pr} a Banach space?

Hint: For the completeness, prove and apply the Fatou property: if $0\leqslant f_n\uparrow f$ and $\sup_n\|f_n\|_{L_{pr}}$ is finite then $f\in L_{pr}$ and $\lim_{n\to\infty}\|f_n\|_{L_{pr}}=\|f\|_{L_{pr}}$.

To prove the triangle inequality, note that formula (5) in Section 1.3 is equivalent to the formula

$$\int_0^t f^*(x)dx=\sup_{m(E)\leqslant t}\int_E|f(x)|dx, \qquad t>0$$

and thus, with $h\in L^\infty$ non-negative and decreasing, that

$$\int_0^t h(x)(f+g)^*(x)\,dx\leqslant\int_0^t h(x)\,f^*(x)dx+\int_0^t h(x)\,g^*(x)\,dx, \qquad t>0,$$

holds.

7. (O'Neil [1]). Prove that if $1<p<\infty$, $1<r<\infty$, and $f\in L_{pr}$ then, with

$$f^{**}(t)=t^{-1}\int_0^t f^*(s)\,ds, \qquad t>0,$$

$$\|f\|'_{L_{pr}}=\|f^{**}\|_{L_{pr}},$$

$\|\cdot\|'_{L_{pr}}$ is a norm on L_{pr} and $\|f\|'_{L_{pr}}\sim\|f\|_{L_{pr}}$ (i.e., there exist positive constants C_1, C_2 such that the inequalities $C_1\|f\|_{L_{pr}}\leqslant\|f\|'_{L_{pr}}\leqslant C_2\|f\|_{L_{pr}}$ hold for all $f\in L_{pr}$).

Hint: Apply Hardy's inequality (see Hardy *et al.* [1])

$$\int_0^\infty(t^{1/p}\cdot t^{-1}\int_0^t f^*(s)ds)^r\,dt/t\leqslant C_{pr}\int_0^\infty(t^{1/p}f^*(t))^r\,dt/t\,.$$

8. (Lorentz [2]). Show that if $1\leqslant r_1<r_2\leqslant\infty$ and $1<p<\infty$ then

$$L_{pr_1}\subset L_{pr_2}\,.$$

Hint: Prove first the result for $r_2=\infty$. To this end, note that

$$t^{1/p}f^*(t)\leqslant C_p\int_0^t s^{1/p}f^*(s)ds/s\,, \qquad t>0\,.$$

9. (Hunt [1]). Prove that the restriction $p\leqslant q$ in the Marcinkiewicz theorem is indispensable.

Hint: Consider $(0,\infty)$ with Lebesgue measure. Put

$$Tf(x)=x^{-\alpha-1}\int_0^x f(t)dt, \qquad \alpha>0,$$

and verify that

$$(Tf)^*(x) \leqslant x^{-\alpha} f^{**}(x),$$
$$(Tf)^*(x) \geqslant x^{-\alpha} f^*(x) \quad \text{if} \quad f \equiv f^*.$$

Choose $\alpha = 1/q_i - 1/p_i$, $i=0,1$, and use the results in the two previous exercises to show that there is a function $f \in L_p$ for which $Tf \notin L_q$, where $p>q$ are chosen as in the Marcinkiewicz theorem.

10. Prove that if $1 \leqslant p \leqslant 2$, $1/p' = 1 - 1/p$ and $p \leqslant q \leqslant p'$ then

$$\|\mathscr{F} f\|_{L_p(|\xi|^{-n\rho})} \leqslant C_{pq} \|f\|_{L_p}$$

where $\rho = 1 - q/p'$.

Hint: Apply 1.2.1 and 1.4.1.

11. (Stein [1]). Consider a family of operators T_z, such that $T_z f$ is a vector-valued analytic function of z for $0 < \operatorname{Re} z < 1$ and continuous for $0 \leqslant \operatorname{Re} z \leqslant 1$ for each fixed f in the domain. Prove that if

$$T_{iy}: \quad L_{p_0} \to L_{q_0},$$
$$T_{1+iy}: L_{p_1} \to L_{q_1},$$

with $\max(\log\|T_{iy}\|, \log\|T_{1+iy}\|) \leqslant Ce^{a|y|}$, $0 < a < \pi$, then

$$T_\theta: L_p \to L_q$$

with $\|T_\theta\| \leqslant h(\theta, \|T_{i\cdot}\|, \|T_{1+i\cdot}\|)$ where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, $0 \leqslant \theta \leqslant 1$, and h is bounded in $0 < \theta < 1$ for fixed T_z. ($\|T_{i\cdot}\|$ denotes the function.)

Hint: Adapt the proof of Theorem 1.1.1, using a conformal mapping.

12. (Stein-Weiss [1]). Assume that

$$T: L_{p_0}(U, d\mu_0) \to L_{q_0}(V, d\nu_0),$$
$$T: L_{p_1}(U, d\mu_1) \to L_{q_1}(V, d\nu_1).$$

Then show that

$$T: L_p(U, d\mu) \to L_q(V, d\nu),$$

with norm

$$M \leqslant M_0^{1-\theta} M_1^\theta,$$

provided that

$$1/p = (1-\theta)/p_0 + \theta/p_1, \qquad 1/q = (1-\theta)/q_0 + \theta/q_1$$

and

$$\mu=\mu_0^{p(1-\theta)/p_0}\mu_1^{p\theta/p_1}, \qquad \nu=\nu_0^{p(1-\theta)/p_0}\nu_1^{p\theta/p_1}.$$

The last two equations mean that μ_0 and μ_1 are both absolutely continuous with respect to a measure σ, i.e., $\mu_0=\omega_0\sigma$, $\mu_1=\omega_1\sigma$ and $\mu=\omega_0^{p(1-\theta)/p_0}\omega_1^{p\theta/p_1}\sigma$. Similarly for ν_0, ν_1 and ν.

Hint: Use the proof of the Riesz-Thorin theorem, and put

$$\varphi(z)=\varphi(x,z)=|f(x)|^{p/p(z)}\frac{f(x)}{|f(x)|}\,\omega(x)^{1/p(z)}\omega_0(x)^{-(1-z)/p_0}\omega_1(x)^{-z/p_1},$$

and choose $\psi(z)$ analogously.

13. (Thorin [2]). Assume that (with $L_p=L_p(U,d\mu)$)

$$T\colon L_{p_1^{(0)}}\times L_{p_2^{(0)}}\times\cdots\times L_{p_n^{(0)}}\to L_{q_0},$$
$$T\colon L_{p_1^{(1)}}\times L_{p_2^{(1)}}\times\cdots\times L_{p_n^{(1)}}\to L_{q_1}.$$

Then show that

$$T\colon L_{p_1}\times L_{p_2}\times\cdots\times L_{p_n}\to L_q,$$

with norm

$$M\leqslant M_0^{1-\theta}M_1^{\theta}$$

where $1/p_i=(1-\theta)/p_i^{(0)}+\theta/p_i^{(1)}$, $1/q=(1-\theta)/q_0+\theta/q_1$, and $0\leqslant\theta\leqslant1$.

Hint: Adapt the proof of Theorem 1.1.1.

14. (Salem and Zygmund [1]). Let f be holomorphic in the open unit disc and $0<p\leqslant\infty$.

Then we write $f\in H_p$ (Hardy class) if the expression

$$\|f\|_{H_p}=\sup_{0<r<1}\left(\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p}$$

is finite. Show that if

$$T\colon H_{p_0}\to L_{q_0},$$
$$T\colon H_{p_1}\to L_{q_1}$$

then

$$T\colon H_p\to L_q$$

with norm

$$M\leqslant M_0^{1-\theta}M_1^{\theta}$$

where $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$, $0\leqslant\theta\leqslant 1$, $0<p_0, p_1\leqslant\infty$, $1\leqslant q_0$, $q_1\leqslant\infty$.

Hint: Note the fact, due to F. Riesz, that every $f\in H_p$ admits a factorization $f=Bg$, where B is a Blaschke product with the same zeros as f, and $g\in H_p$ has no zeros in the disc. Moreover, reformulate Exercise 4 as follows: If $p>1$ then H_p is a complemented subspace of L_p. Write π for the corresponding projection and consider the mapping

$$F: L_{p_1}\times\cdots\times L_{p_n}\to H_p$$

where $1/p=\sum_{i=1}^n 1/p_i$ and $p_i>1$, $i=1,2,\ldots,n$, defined by

$$F(f_1,\ldots,f_n)\to\pi f_1\cdot\pi f_2\ldots\pi f_n$$

Obviously, by F. Riesz, $f=\varphi_1\cdot\cdots\cdot\varphi_n$, $\varphi_i\in H_{p_i}$, and so $f=F(\varphi_1,\ldots,\varphi_n)$. Apply Exercise 12 to the mapping $M=T\circ F$.

15. Write Kolmogorov's [1] inequality

$$\sup_{x\in\mathbf{R}}|D^1a(x)|\leqslant C(\sup_{x\in\mathbf{R}}|a(x)|)^{1-1/m}(\sup_{x\in\mathbf{R}}|D^m a(x)|)^{1/m},\qquad 0\leqslant 1\leqslant m,$$

where a is m-times continuously differentiable, in the form indicated in Section 1.5.

1.7. Notes and Comment

1.7.1. An early instance of interpolation of linear operators, due to I. Schur [1] in 1911, is reproduced as Exercise 1. He stated his result for bilinear forms, or rather, for the matrices corresponding to the forms.

In 1926, M. Riesz [1] proved the first version of the Riesz-Thorin theorem with the restriction $p\leqslant q$, which he showed is essential when the scalars are real. Riesz's main tool was the Hölder inequality. These early results were given for bilinear forms and l_p, but they have equivalent versions in the form of the theorems in the text, cf. Hardy, Littlewood and Pólya [1]. Giving an entirely new proof, G. O. Thorin [1] in 1938 was able to remove the restriction $p\leqslant q$. Thorin used complex scalars and the maximum principle whereas Riesz had real scalars and Hölder's inequality. Moreover, Thorin gave a multilinear version of the theorem (see Exercise 13). A generalization to sublinear operators was given by Calderón and Zygmund [1], another by Stein [1], and yet another with change of measures by Stein and Weiss [1]. The latter two generalizations are found in Exercises 11 and 12. Finally, Krée [1] has given an extension to $p<1$, $q<1$, i.e., the quasi-normed case. Other proofs and extensions have been given by several authors (for references see Zygmund [1]).

We reconsider the Riesz-Thorin theorem in Chapter 4 and Chapter 5, and then in a general framework.

1.7.2. The Hausdorff-Young inequality (Theorem 1.2.1) is a generalization of Parseval's theorem and the Riemann-Lebesgue lemma. (There is also an inverse version using the Riesz-Fischer theorem; see Zygmund [1].) It was first obtained on the torus $\mathbb{T}$ by W.H. Young [1] in 1912 for p' even, and then, in 1923, for general p by F. Hausdorff [1]. Young employed his inequality, Theorem 1.2.2, given for bilinear forms, which he proved by a repeated application of Hölder's inequality. There are examples (e.g., in Zygmund [1] for the torus $\mathbb{T}$) which show that the condition $p \leqslant 2$ is essential in the Hausdorff-Young theorem. F. Riesz [1] in 1923 proved an analogue of the Hausdorff-Young theorem for any orthogonal system. This is Exercise 2, where the idea to use interpolation for the proof appeared in M. Riesz [1]. A further extension of the Hausdorff-Young inequality to locally compact Abelian groups has been made by Weil [1]. His proof is quite analogous to the one given in the text. This proof, using interpolation directly, is due to M. Riesz [1]. Another generalization is discussed in the Notes and Comment pertaining to Section 1.4.

The space m_p of Fourier multipliers (Exercise 3—5) has been treated in Hörmander [1] and Larsen [1]. The Fourier multipliers are our main tools in Chapter 6, treating the Sobolev and Besov spaces.

The use of the Riesz-Thorin theorem to obtain results about the Hardy classes H_p (Exercise 14) was introduced by Thorin [2] and Salem and Zygmund [1]. We return to H_p in Chapter 6.

Results for the trace classes $\mathfrak{S}_p$ of compact operators in a Hilbert space have been proved analogously to the L_p case by an extension of the results to non-commutative integration, compare, for example, Gohberg-Krein [1] and Peetre-Sparr [2].

1.7.3. The Marcinkiewicz theorem appeared in a note by J. Marcinkiewicz [1] in 1939, without proof. A. Zygmund [2] in 1956 gave a proof (using distribution functions) and also applications of the theorem, which can not be obtained by the Riesz-Thorin result. Independently, Cotlar [1] has given a similar proof. The condition $p \leqslant q$ is essential; this was shown by R.A. Hunt [1] in 1964, cf. Exercise 9. Several extensions have been given. A.P. Calderón [3] gave a version for general Lorentz spaces and quasi-linear operators, viz.,

$$|T(\lambda f)(x)| \leqslant k_1 |\lambda| \, |Tf(x)|,$$

$$|T(f+g)(x)| \leqslant k_2(|Tf(x)| + |Tf(x)|),$$

where k_1 and k_2 are constants. It is not hard to see that the proof given in the text works for quasi-linear operators too. Calderón's version has been complemented by Hunt [1]. We return to this topic in Chapter 5. See also Sargent [1], Steigerwalt-White [1], Krein-Semenov [1] and Berenstein *et al.* [1].

Cotlar and Bruschi [1] have shown that the Riesz-Thorin theorem, with the restriction $p \leqslant q$, follows from the Marcinkiewicz theorem, although without the sharp norm inequality.

The proof in the text of the Marcinkiewicz theorem is due to Bergh. The inequality (8) seems to be new. The present proof of this inequality, using discretization, is due to Peetre (personal communication).

The Lorentz spaces were introduced by G.G. Lorentz [1] in 1950. Later he generalized his ideas, e.g. in [2], where the present definition may be found. Our notation is due to R. O'Neil [1] and Calderón [3]. In general, the Lorentz spaces are only quasi-normed, but they may be equipped with equivalent norms. For the details, see Exercises 6—8. A still more general type of spaces, Banach function spaces, has been treated by W.A.J. Luxemburg [1] and by Luxemburg and Zaanen [1]. More about the Lorentz spaces is found in the Notes and Comment in Chapter 5.

1.7.4. R.E.A.C. Paley's [1] sharpening of the Hausdorff-Young theorem appeared in 1931. It has been complemented by a sharpening of Young's inequality due to O'Neil [1]. We deal with these questions in Chapter 5. Some of the most important applications of the Marcinkiewicz theorem are those concerning the Hilbert transform and the potential operator. These applications are treated in Chapter 6 and Chapter 5 respectively.

1.7.5. The Bernstein inequality was obtained by Bernstein [1] in 1912, and the Jackson inequality by Jackson [1] in 1912. Cf. Lorentz [3].

Interpolation of linear operators has been used to prove results about approximation of functions, of operators and, in particular, of differential operators by difference operators. (See Peetre-Sparr [1] and Löfström [1] as general references.) Chapter 7 is devoted to these questions, and we refer to this chapter for precise statements and references.

Chapter 2

General Properties of Interpolation Spaces

In this chapter we introduce some basic notation and definitions. We discuss a few general results on interpolation spaces. The most important one is the Aronszajn-Gagliardo theorem.

This theorem says, loosely speaking, that if a Banach space A is an interpolation space with respect to a Banach couple (A_0, A_1) (of Banach spaces), then there is an interpolation method (functor), which assigns the space A to the couple (A_0, A_1).

2.1. Categories and Functors

In this section we summarize some general notions, which will be used in what follows. A more detailed account can be found, for instance, in MacLane [1].

A *category* $\mathscr{C}$ consists of *objects* $A, B, C, \ldots$ and *morphisms* $R, S, T, \ldots$. Between objects and morphisms a three place relation is defined, $T: A \curvearrowright B$. If $T: A \curvearrowright B$ and $S: B \curvearrowright C$ then there is a morphism ST, the product of S and T, such that $ST: A \curvearrowright C$. The product of morphisms satisfies the associative law

$$(1) \qquad T(SR) = (TS)R.$$

Moreover, for any object A in $\mathscr{C}$, there is a morphism $I = I_A$, such that for all morphisms $T: A \curvearrowright A$ we have

$$(2) \qquad TI = IT = T.$$

In this book we shall frequently work with categories of topological spaces. Thus the objects are certain topological spaces. The morphisms are continuous mappings, ST is the composite mapping, I is the identity. Usually, morphisms are structure preserving mappings. For instance, in the category of all topological vector spaces we take as morphisms all continuous linear operators.

Let $\mathscr{C}_1$ and $\mathscr{C}$ be any two categories. By a *functor* F from $\mathscr{C}_1$ into $\mathscr{C}$, we mean a rule which to every object A in $\mathscr{C}_1$ assigns an object $F(A)$ in $\mathscr{C}$, to every morphism

T in $\mathscr{C}_1$ there corresponds a morphism $F(T)$ in $\mathscr{C}$. If $T: A \curvearrowright B$ then $F(T): F(A) \curvearrowright F(B)$ and

(3) $$F(ST) = F(S)F(T),$$

(4) $$F(I_A) = I_{F(A)}.$$

Note that our concept "functor" is usually called "covariant functor".

As a simple example, let $\mathscr{C}$ be the category of all topological vector spaces and $\mathscr{C}_1$ the category of all finite dimensional Euclidean spaces. The morphisms are the continuous linear operators. Now define $F(A) = A$ and $F(T) = T$. Then F is of course a functor from $\mathscr{C}_1$ into $\mathscr{C}$.

In general, let $\mathscr{C}$ and $\mathscr{C}_1$ be two categories, such that every object in $\mathscr{C}_1$ is an object in $\mathscr{C}$ and every morphism in $\mathscr{C}_1$ is a morphism in $\mathscr{C}$. Then we say that $\mathscr{C}_1$ is a *sub-category* of $\mathscr{C}$ if $F(A) = A$ and $F(T) = T$ defines a functor from $\mathscr{C}_1$ to $\mathscr{C}$.

2.2. Normed Vector Spaces

In this section we introduce some of the categories of topological vector spaces which we shall use frequently.

Let A be a vector space over the real or complex field. Then A is called a *normed vector space* if there is a real-valued function (a norm) $\|\cdot\|_A$ defined on A such that

(1) $$\|a\|_A \geqslant 0, \quad \text{and} \quad \|a\|_A = 0 \quad \text{iff} \quad a = 0,$$

(2) $$\|\lambda a\|_A = |\lambda| \, \|a\|_A, \qquad \lambda \text{ a scalar},$$

(3) $$\|a+b\|_A \leqslant \|a\|_A + \|b\|_A.$$

If A is a normed vector space there is a natural topology on A. A neighbourhood of a consists of all b in A such that $\|b-a\|_A < \varepsilon$ for some fixed $\varepsilon > 0$.

Let A and B be two normed vector spaces. Then a mapping T from A to B is called a *bounded linear operator* if $T(\lambda a) = \lambda T(a)$, $T(a+b) = T(a) + T(b)$ and if

$$\|T\|_{A,B} = \sup_{a \neq 0} \|Ta\|_B / \|a\|_A < \infty .$$

Clearly any bounded linear operator is continuous. The space of all bounded linear operators from A to B is a new normed vector space with norm $\|\cdot\|_{A,B}$.

We shall reserve the letter $\mathscr{N}$ to denote the category of all normed vector spaces. The objects of $\mathscr{N}$ are normed vector spaces and the morphisms are the bounded linear operators. Thus $\mathscr{N}$ is a sub-category of the category of all topological vector spaces.

A natural sub-category of $\mathcal{N}$ is the category of complete normed vector spaces or *Banach spaces*. Recall that a normed vector space A is called *complete* if every *Cauchy sequence* $(a_n)_1^\infty$ in A has a limit in A, i.e. if the condition

$$\|a_n - a_m\|_A \to 0 \quad \text{as} \quad \min(n,m) \to \infty,$$

implies the existence of an element $a \in A$, such that

$$\|a_n - a\|_A \to 0 \quad \text{as} \quad n \to \infty.$$

In many cases it is preferable to prove completeness by means of the following "absolute convergence implies convergence" test.

2.2.1. Lemma. *Suppose that A is a normed vector space. Then A is complete if and only if*

$$\sum_{n=1}^\infty \|a_n\|_A < \infty$$

implies that there is an element $a \in A$ such that

$$\|a - \sum_{n=1}^N a_n\|_A \to 0 \quad \textit{as} \quad N \to \infty.$$

Proof: Suppose first that A is complete and that $\sum \|a_n\|_A$ converges. Clearly $(\sum_{n=1}^\nu a_n)$ is then a Cauchy sequence in A and thus $a = \sum_1^\infty a_n$ with convergence in A. For the other implication, suppose that (a_ν) is a Cauchy sequence in A. It is easy to see that we may choose a subsequence (a_{ν_j}) with $\sum_{j=1}^\infty \|a_{\nu_j} - a_{\nu_{j-1}}\|_A$ finite. Then it follows that $\sum_{j=1}^\infty (a_{\nu_j} - a_{\nu_{j-1}})$ converges in A and thus (a_{ν_j}) converges in A too. But then (a_ν) also converges in A since it is a Cauchy sequence. □

We shall use the letter $\mathcal{B}$ to denote the category of all Banach spaces. Thus $\mathcal{B}$ is a sub-category of $\mathcal{N}$. Other familiar sub-categories of $\mathcal{N}$ are the category of all Hilbert spaces (which is also a sub-category of $\mathcal{B}$) and the category of all finite dimensional Euclidean spaces.

2.3. Couples of Spaces

Let A_0 and A_1 be two topological vector spaces. Then we shall say that A_0 and A_1 are *compatible* if there is a Hausdorff topological vector space $\mathfrak{A}$ such that A_0 and A_1 are sub-spaces of $\mathfrak{A}$. Then we can form their sum $A_0 + A_1$ and their intersection $A_0 \cap A_1$. The sum consists of all $a \in \mathfrak{A}$ such that we can write $a = a_0 + a_1$ for some $a_0 \in A_0$ and $a_1 \in A_1$.

2.3.1. Lemma. *Suppose that A_0 and A_1 are compatible normed vector spaces. Then $A_0 \cap A_1$ is a normed vector space with norm defined by*

(1) $$\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1}).$$

Moreover, A_0+A_1 *is also a normed vector space with norm*

(2) $$\|a\|_{A_0+A_1}=\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+\|a_1\|_{A_1}).$$

If A_0 *and* A_1 *are complete then* $A_0\cap A_1$ *and* A_0+A_1 *are also complete.*

Proof: The proof is straightforward. We shall only give the proof of the completeness of A_0+A_1. We use Lemma 2.2.1. Assume that

$$\sum_{n=1}^{\infty}\|a_n\|_{A_0+A_1}<\infty.$$

Then we can find a decomposition $a_n=a_n^0+a_n^1$, such that

$$\|a_n^0\|_{A_0}+\|a_n^1\|_{A_1}\leqslant 2\,\|a_n\|_{A_0+A_1}.$$

It follows that

$$\sum_{n=1}^{\infty}\|a_n^0\|_{A_0}<\infty,\qquad \sum_{n=1}^{\infty}\|a_n^1\|_{A_1}<\infty.$$

If A_0 and A_1 are complete we obtain from Lemma 2.2.1. that $\sum a_n^0$ converges in A_0 and $\sum a_n^1$ converges in A_1. Put $a^0=\sum a_n^0$ and $a^1=\sum a_n^1$ and $a=a^0+a^1$. Then $a\in A_0+A_1$ and since

$$\|a-\sum_{n=1}^{N}a_n\|_{A_0+A_1}\leqslant\|a^0-\sum_{n=1}^{N}a_n^0\|_{A_0}+\|a^1-\sum_{n=1}^{N}a_n^1\|_{A_0}$$

we conclude that $\sum_n a_n$ converges in A_0+A_1 to a. □

Let $\mathscr{C}$ denote any sub-category of the category $\mathscr{N}$ of all normed vector spaces. We assume that the mappings $T\colon A\to B$ are all bounded linear operators from A to B. We let $\mathscr{C}_1$ stand for a category of *compatible couples* $\bar{A}=(A_0,A_1)$, i.e. such that A_0 and A_1 are compatible and such that A_0+A_1 and $A_0\cap A_1$ are spaces in $\mathscr{C}$. The morphisms $T\colon(A_0,A_1)\to(B_0,B_1)$ in $\mathscr{C}_1$ are all bounded linear mappings from A_0+A_1 to B_0+B_1 such that

$$T_{A_0}\colon A_0\to B_0,\quad T_{A_1}\colon A_1\to B_1$$

are morphisms in $\mathscr{C}$. Here T_A denotes the restriction of T to A. With the natural definitions of composite morphism and identity, it is easy to see that $\mathscr{C}_1$ is in fact a category. In the sequel, T will stand for the restrictions to the various subspaces of A_0+A_1. We have, with $a=a_0+a_1$,

$$\|Ta\|_{B_0+B_1}\leqslant\|T\|_{A_0,B_0}\|a_0\|_{A_0}+\|T\|_{A_1,B_1}\|a_1\|_{A_1}.$$

Writing $\|T\|_{A,B}$ for the norm of the mapping $T\colon A\to B$, we conclude

(3) $$\|T\|_{A_0+A_1,B_0+B_1}\leqslant\max(\|T\|_{A_0,B_0},\|T\|_{A_1,B_1}),$$

and

(4) $$\|T\|_{A_0\cap A_1, B_0\cap B_1} \leqslant \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}).$$

We can define two basic functors Σ (*sum*) and Δ (*intersection*) from $\mathscr{C}_1$ to $\mathscr{C}$. We write $\Sigma(T)=\Delta(T)=T$ and

(5) $$\Delta(\bar{A})=A_0\cap A_1,$$

(6) $$\Sigma(\bar{A})=A_0+A_1.$$

As a simple example we take $\mathscr{C}=\mathscr{B}$. By Lemma 2.3.1 we can take as $\mathscr{C}_1$ all compatible couples (A_0, A_1) of Banach spaces. In fact, Lemma 2.3.1 implies that if A_0 and A_1 are compatible, then A_0+A_1 and $A_0\cap A_1$ are Banach spaces. As a second example we take the category $\mathscr{C}$ of all spaces $L_{1,w}$ defined by the norms

$$\|f\|_{L_{1,w}}=\int|f(x)|\,w(x)\,dx$$

where $w(x)>0$. Since $L_{1,w_0}\cap L_{1,w_1}=L_{1,w'}$ where $w'(x)=\max(w_0(x), w_1(x))$, and since $L_{1,w_0}+L_{1,w_1}=L_{1,w''}$ where $w''(x)=\min(w_0(x), w_1(x))$, we can let $\mathscr{C}_1$ consist of all couples (L_{1,w_0}, L_{1,w_1}).

As a third example we consider the category $\mathscr{C}$ of all Banach algebras (Banach spaces with a continuous multiplication). $\mathscr{C}_1$ consists of all compatible couples (A_0, A_1) such that A_0 and A_1 are Banach algebras with the same multiplication and such that A_0+A_1 is a Banach algebra with that multiplication. Since it is easily seen that $A_0\cap A_1$ is also a Banach algebra, we conclude that $\mathscr{C}_1$ satisfies the requirements listed above. (Note that A_0+A_1 is not in general a Banach algebra.)

In most cases we shall deal with the categories $\mathscr{C}=\mathscr{N}$ or $\mathscr{C}=\mathscr{B}$. Then $\mathscr{C}_1$ will denote the category of all compatible couples of spaces in $\mathscr{C}$. This will be our general convention. If $\mathscr{C}$ is any given category, which is closed under the operations Σ and Δ, then $\mathscr{C}_1$ denotes the category of all compatible couples.

2.4. Definition of Interpolation Spaces

In this section $\mathscr{C}$ denotes any sub-category of the category $\mathscr{N}$, such that $\mathscr{C}$ is closed under the operations sum and intersection. We let $\mathscr{C}_1$ stand for the category of all compatible couples $\bar{A}$ of spaces in $\mathscr{C}$.

2.4.1. Definition. Let $\bar{A}=(A_0, A_1)$ be a given couple in $\mathscr{C}_1$. Then a space A in $\mathscr{C}$ will be called an *intermediate space* between A_0 and A_1 (or with respect to $\bar{A}$) if

(1) $$\Delta(\bar{A})\subset A\subset\Sigma(\bar{A})$$

with continuous inclusions. The space A is called an *interpolation space* between A_0 and A_1 (or with respect to $\bar{A}$) if in addition

(2) $\qquad T: \bar{A} \to \bar{A} \quad \text{implies} \quad T: A \to A.$

More generally, let $\bar{A}$ and $\bar{B}$ be two couples in $\mathscr{C}_1$. Then we say that two spaces A and B in $\mathscr{C}$ are *interpolation spaces* with respect to $\bar{A}$ and $\bar{B}$ if A and B are intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively, and if

(3) $\qquad T: \bar{A} \to \bar{B} \quad \text{implies} \quad T: A \to B. \quad \square$

To avoid a possible misunderstanding, we remark here that if A and B are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$, then it does not, in general, follow that A is an interpolation space with respect to $\bar{A}$, or that B is an interpolation space with respect to $\bar{B}$. (See Section 2.9.)

Note that (3) means that if $T: A_0 \to B_0$ and $T: A_1 \to B_1$ then $T: A \to B$. Thus (2) and (3) are the interpolation properties we have already met in Chapter 1. As an example, the Riesz-Thorin theorem shows that L_p is an interpolation space between L_{p_0} and L_{p_1} if $p_0 < p < p_1$.

Clearly $\Delta(\bar{A})$ and $\Delta(\bar{B})$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. The same is true for $\Sigma(\bar{A})$ and $\Sigma(\bar{B})$. If $A = \Delta(\bar{A})$ (or $\Sigma(\bar{A})$) and $B = \Delta(\bar{B})$ (or $\Sigma(\bar{B})$), then we have

(4) $\qquad \|T\|_{A,B} \leqslant \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}).$

(See Section 2.3, Formula (3) and (4).)

In general, if (4) holds we shall say that A and B are *exact interpolation spaces.* In many cases it is only possible to prove

(5) $\qquad \|T\|_{A,B} \leqslant C \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}).$

Then we shall say that A and B are *uniform interpolation spaces.* In fact, it follows from Theorem 2.4.2 below that, when $B, B_i, i=0,1,$ are complete, A and B are interpolation spaces iff (5) holds, i.e., (3) and (5) are then equivalent. Also, (2) and (5) are equivalent for $B = A, B_i = A_i, i = 0, 1,$ when all the spaces are complete.

The interpolation spaces A and B are of *exponent* θ, $(0 \leqslant \theta \leqslant 1)$ if

(6) $\qquad \|T\|_{A,B} \leqslant C \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}.$

If $C=1$ we say that A and B are *exact of exponent* θ.

Note that (6) is a convexity result of the type we have met in Chapter 1. By the Riesz-Thorin theorem, L_p is an interpolation space between L_{p_0} and L_{p_1} which is exact of exponent θ, if

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad (0 < \theta < 1).$$

Similarly the Marcinkiewicz theorem implies that L_p and L_q are interpolation spaces with respect to (L_{p_0}, L_{p_1}) and $(L^*_{q_0}, L^*_{q_1})$. Here L_p and L_q are interpolation spaces of exponent θ (not exact), if

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}, \qquad (0<\theta<1).$$

We shall now discuss some simple properties of interpolation spaces.

2.4.2. Theorem. *Consider the category $\mathscr{B}$. Suppose that A and B are interpolation spaces with respect to the couples $\bar{A}$ and $\bar{B}$. Then A and B are uniform interpolation spaces.*

Proof: Consider the set of all morphisms T in $\mathscr{C}_1$ such that $T: \bar{A} \to \bar{B}$. Thus T is also bounded and linear from A to B. Denote this set equipped with the norm $\max(\|T\|_{A,B}, \|T\|_{A_0,B_0}, \|T\|_{A_1,B_1})$ by L_1, and equipped with the norm $\max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1})$ by L_2. It is easily verified that L_1 and L_2 are Banach spaces. (Use the intermediate space properties.) The identity mapping $i: L_1 \to L_2$ is clearly linear, bounded and bijective. By the Banach theorem $i^{-1}: L_2 \to L_1$ is also bounded. This means that we have $\|T\|_{A,B} \leqslant \max(\|T\|_{A,B}, \|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}) \leqslant C \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1})$, with C independent of T, i.e. (5) holds. □

A major objective in interpolation theory is the actual construction of interpolation spaces. A method of constructing such a space will be called an interpolation functor according to the following definition.

2.4.3. Definition. By an *interpolation functor* (or *interpolation method*) on $\mathscr{C}$ we mean a functor F from $\mathscr{C}_1$ into $\mathscr{C}$ such that if $\bar{A}$ and $\bar{B}$ are couples in $\mathscr{C}_1$, then $F(\bar{A})$ and $F(\bar{B})$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. Moreover we shall have

$$F(T)=T \quad \text{for all} \quad T: \bar{A} \to \bar{B}. \quad \square$$

We shall say that F is a *uniform (exact) interpolation functor* if $F(\bar{A})$ and $F(\bar{B})$ are uniform (exact) interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. Similarly we say that F is *(exact) of exponent* θ if $F(\bar{A})$ and $F(\bar{B})$ are (exact) of exponent θ.

By Theorem 2.4.2, any interpolation functor F on $\mathscr{B}$ is uniform. Note that this means that

$$\|T\|_{F(\bar{A}),F(\bar{B})} \leqslant C \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}),$$

for some constant C depending on the couples $\bar{A}$ and $\bar{B}$. If we can choose C independent of $\bar{A}$ and $\bar{B}$, we speak of a *bounded interpolation functor*. Note that F is exact if we can take $C=1$.

The simplest interpolation functors are the functors Δ and Σ. These functors are exact interpolation functors on any admissible sub-category $\mathscr{C}$ of the category $\mathscr{N}$ of normed vector spaces.

2.5. The Aronszajn-Gagliardo Theorem

Let A be an interpolation space with respect to $\bar{A}$. It is natural to ask if there is an interpolation functor F, such that $F(\bar{A})=A$. This question is considered in the following theorem.

2.5.1. Theorem (The Aronszajn-Gagliardo theorem). *Consider the category $\mathscr{B}$ of all Banach spaces. Let A be an interpolation space with respect to the couple $\bar{A}$. Then there exists an exact interpolation functor F_0 on $\mathscr{B}$ such that $F_0(\bar{A})=A$.*

Note that $F_0(\bar{A})=A$ means that the spaces $F_0(\bar{A})$ and A have the same elements and equivalent norms. Thus it follows from the theorem that any interpolation space can be renormed in such a way that the renormed space becomes an exact interpolation space.

Proof: Let $\bar{X}=(X_0, X_1)$ be a given couple in $\mathscr{B}_1$. If $T\colon \bar{A}\to\bar{X}$ we write

$$\|T\|_{\bar{A},\bar{X}}=\max(\|T\|_{A_0,X_0}, \|T\|_{A_1,X_1}).$$

Then $X=F_0(\bar{X})$ consists of those $x\in\Sigma(\bar{X})$, which admit a representation

$$x=\textstyle\sum_j T_j a_j \quad \text{(convergence in } \Sigma(\bar{X})),$$

where $T_j\colon \bar{A}\to\bar{X}$, $a_j\in A$. Put

$$N_X(x)=\textstyle\sum_j \|T_j\|_{\bar{A},\bar{X}}\,\|a_j\|_A\,.$$

The norm in X is the infimum of $N_X(x)$ over all admissible representations of x.

First we prove that X is an intermediate space with respect to $\bar{X}$. In order to prove that $\Delta(\bar{X})\subset X$ we let φ be a bounded linear functional on $\Sigma(\bar{A})$ such that $\varphi(a_1)=1$ for some fixed $a_1\in A$. Let $x\in\Delta(\bar{X})$ be fixed and put $T_1 a=\varphi(a)x$. Then

$$\|T_1 a\|_{X_j}=|\varphi(a)|\,\|x\|_{X_j}\leqslant C\,\|a\|_{\Sigma(\bar{A})}\,\|x\|_{X_j}\,.$$

Since $A_j\subset\Sigma(\bar{A})$ we conclude that $T_1\colon\bar{A}\to\bar{X}$ and

$$\|T_1\|_{\bar{A},\bar{X}}\leqslant C\,\|x\|_{\Delta(\bar{X})}.$$

Put $T_j=0$ and $a_j=0$ if $j>1$. Since $T_1 a_1=x$ we then have $x=\sum_j T_j a_j$ and

$$\|x\|_X\leqslant\textstyle\sum_j \|T_j\|_{\bar{A},\bar{X}}\,\|a_j\|_A\leqslant C\,\|x\|_{\Delta(\bar{X})}\,\|a_1\|_A\,.$$

This implies $\Delta(\bar{X})\subset X$. The inclusion $X\subset\Sigma(\bar{X})$ follows easily from the fact that $A\subset\Sigma(\bar{A})$. For if $x=\sum_j T_j a_j$ is an admissible representation of $x\in X$, then

$$\|x\|_{\Sigma(\bar{X})}\leqslant\textstyle\sum_j \|T_j\|_{\bar{A},\bar{X}}\,\|a_j\|_{\Sigma(\bar{A})}\leqslant C\sum_j \|T_j\|_{\bar{A},\bar{X}}\,\|a_j\|_A\,.$$

Thus

$$\|x\|_{\Sigma(\bar{X})} \leqslant C N_X(x)$$

which implies $X \subset \Sigma(\bar{X})$.

We now turn to the completeness of X, using Lemma 2.2.1 repeatedly. Suppose that $\sum_{v=0}^{\infty} \|x^{(v)}\|_X$ converges. Then $\sum_v \|x^{(v)}\|_{\Sigma(\bar{X})}$ converges too, since $X \subset \Sigma(\bar{X})$. Thus $x = \sum x^{(v)}$ with convergence in $\Sigma(\bar{X})$, $\Sigma(\bar{X})$ being complete. Let $x^{(v)} = \sum T_j^{(v)} a_j^{(v)}$ be admissible representations such that $\sum \|T_j^{(v)}\|_{\bar{A},\bar{X}} \|a_j^{(v)}\|_A < \|x^{(v)}\|_X + 2^{-v}$, $v = 0, 1, 2, \ldots$. Then $x = \sum_v \sum_j T_j^{(v)} a_j^{(v)}$ is in X because $\sum_v \sum_j \|T_j^{(v)}\|_{\bar{A},\bar{X}} \|a_j^{(v)}\|_A < \infty$. Finally, with these representations, we have

$$\begin{aligned} \|x - \textstyle\sum_0^n x^{(v)}\|_X &\leqslant \textstyle\sum_{v=n+1}^{\infty} \sum_{j=0}^{\infty} \|T_j^{(v)}\|_{\bar{A},\bar{X}} \|a_j^{(v)}\|_A \\ &\leqslant \textstyle\sum_{v=n+1}^{\infty} (\|x^{(v)}\|_X + 2^{-v}) \to 0 \quad (n \to \infty). \end{aligned}$$

Thus $x = \sum x^{(v)}$ with convergence in X, and X is complete.

Next we prove that F_0 is an exact interpolation functor. Assume that $S: \bar{X} \to \bar{Y}$. If $\bar{X} = (X_0, X_1)$ and $\bar{Y} = (Y_0, Y_1)$ we write

$$M_j = \|S\|_{X_j, Y_j}, \quad j = 0, 1 .$$

Put $X = F(\bar{X})$ and $Y = F(\bar{Y})$ and suppose that $x \in X$. If $x = \sum_j T_j a_j$ is an admissible representation of x, then $Sx = \sum_j S T_j a_j$ is an admissible representation of Sx. In fact,

$$\|S T_j\|_{\bar{A},\bar{Y}} \leqslant \max(M_0, M_1) \|T_j\|_{\bar{A},\bar{X}}$$

and therefore

$$\textstyle\sum_j \|S T_j\|_{\bar{A},\bar{Y}} \|a_j\|_A \leqslant \max(M_0, M_1) \sum_j \|T_j\|_{\bar{A},\bar{X}} \|a_j\|_A .$$

This proves that $\|Sx\|_Y \leqslant \max(M_0, M_1) \|x\|_X$, i.e., that F_0 is an exact interpolation functor.

It remains to prove that $F_0(\bar{A}) = A$. If $a \in F_0(\bar{A})$ has the admissible representation $a = \sum_j T_j a_j$ where $T_j: \bar{A} \to \bar{A}$ then

$$\|T_j a_j\|_A \leqslant C \|T_j\|_{\bar{A},\bar{A}} \|a_j\|_A .$$

This follows from the fact that A is an interpolation space with respect to $\bar{A}$ and that A is uniform according to Theorem 2.4.2. Thus

$$\|a\|_A \leqslant \textstyle\sum_j \|T_j a_j\|_A \leqslant C \sum_j \|T_j\|_{\bar{A},\bar{A}} \|a_j\|_A = C N_A(a) ,$$

which gives $F_0(\bar{A}) \subset A$. The converse inclusion is immediate. For a given $a \in A$, we write $a = \sum_j T_j a_j$, where $T_j = 0$ and $a_j = 0$ for $j > 1$ and $T_1 = I$, $a_1 = a$. Then $\|a\|_{F_0(\bar{A})} \leqslant \sum_j \|T_j\|_{\bar{A},\bar{A}} \|a_j\|_A = \|a\|_A$. $\square$

Let us look back at the proof. Where did we use that A was an interpolation space? Obviously only when proving that $F_0(\bar{A})\subset A$. Thus we conclude that if A is any intermediate space with respect to $\bar{A}$, then there exists an exact interpolation space B with respect to $\bar{A}$, such that $A\subset B$.

Of greater interest is the following corollary of the Aronszajn-Gagliardo theorem. It states that the functor F_0 is minimal among all functors G such that $G(\bar{A})=A$.

2.5.2. Corollary. *Consider the category* $\mathscr{B}$. *Let A be an interpolation space with respect to $\bar{A}$ and let F_0 be the interpolation functor constructed in the proof of Theorem* 2.5.1. *Then* $F_0(\bar{X})\subset G(\bar{X})$ *for all interpolation functors G such that* $G(\bar{A})=A$.

Proof: If $x=\sum_j T_j a_j$ is an admissible representation for $x\in X=F_0(\bar{X})$, then $T_j:\bar{A}\to\bar{X}$. Put $Y=G(\bar{X})$. Since A and Y are uniform interpolation spaces with respect to $\bar{A}$ and $\bar{X}$ it follows that

$$\|T_j a_j\|_Y\leqslant C\|T_j\|_{\bar{A},\bar{X}}\|a_j\|_A,$$

where

$$\|T_j\|_{\bar{A},\bar{X}}=\max(\|T_j\|_{A_0,X_0},\|T_j\|_{A_1,X_1}).$$

Thus

$$\|x\|_Y\leqslant C\sum_j\|T_j\|_{\bar{A},\bar{X}}\|a_j\|_A.$$

By the definition of X it follows that $X\subset Y$, i.e., $F_0(\bar{X})\subset G(\bar{X})$. □

2.6. A Necessary Condition for Interpolation

In this section we consider the category $\mathscr{C}=\mathscr{N}$ of all normed linear spaces. $\mathscr{C}_1$ is the category of all compatible couples.

With $t>0$ fixed, put

$$K(t,a)=K(t,a;\bar{A})=\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+t\|a_1\|_{A_1}),\quad a\in\Sigma(\bar{A}),$$

$$J(t,a)=J(t,a;\bar{A})=\max(\|a\|_{A_0},t\|a\|_{A_1}),\quad a\in\Delta(\bar{A}).$$

These functionals will be used frequently in the sequel. It is easy to see that $K(t,a)$ and $J(t,a)$, $t>0$, are equivalent norms on $\Sigma(\bar{A})$ and $\Delta(\bar{A})$ respectively. (Cf. Chapter 3.)

2.6.1. Theorem. *Let A and B be uniform interpolation spaces with respect to the couples $\bar{A}$ and $\bar{B}$. Then*

$$J(t,b)\leqslant K(t,a)\quad \textit{for some}\quad t,\quad a\in A,$$

implies

$$b \in B, \quad \|b\|_B \leqslant C\|a\|_A.$$

If A and B are exact interpolation spaces the conclusion holds with $C=1$.

The theorem gives a condition on the norms of the interpolation spaces A and B in terms of the norms of the "endpoint" spaces in $\bar{A}$ and $\bar{B}$.

Proof: Let a, b and t be as in the assumption. Consider the linear operator $Tx = f(x)\cdot b$, where f is a linear functional on $\Sigma(\bar{A})$ with $f(a)=1$ and $|f(x)| \leqslant K(t,x)/K(t,a)$. The existence of f follows from the Hahn-Banach theorem. If $x \in A_i$ we have

$$t^i\|Tx\|_i \leqslant |f(x)|\,t^i\|b\|_i \leqslant \frac{K(t,x)}{K(t,a)}\,t^i\|b\|_i \leqslant K(t,x) \leqslant t^i\|x\|_i,$$

$i=0,1$. Hence, since A and B are uniform interpolation spaces, $\|Tx\|_B \leqslant C\|x\|_A$, $x \in A$. Putting $x=a$ we have $\|b\|_B \leqslant C\|a\|_A$ since $Ta=b$. Finally, if A and B are exact, obviously $C=1$. The proof is complete. □

2.7. A Duality Theorem

Considering the category $\mathscr{B}$ of all Banach spaces we have the following.

2.7.1. Theorem. *Suppose that $\Delta(\bar{A})$ is dense in both A_0 and A_1. Then $\Delta(\bar{A})' = \Sigma(\bar{A}')$ and $\Sigma(\bar{A})' = \Delta(\bar{A}')$, where $\bar{A}' = (A_0', A_1')$ and A' denotes the dual of A. More precisely*

$$\|a'\|_{\Sigma(\bar{A}')} = \sup_{a \in \Delta(\bar{A})} \frac{|\langle a', a\rangle|}{\|a\|_{\Delta(\bar{A})}}$$

and

$$\|a'\|_{\Delta(\bar{A}')} = \sup_{a \in \Sigma(\bar{A})} \frac{|\langle a', a\rangle|}{\|a\|_{\Sigma(\bar{A})}}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $\Delta(\bar{A})$ and $\Delta(\bar{A})'$.

Proof: We prove only the first formula. The proof of the second one is quite similar.

First, let $a' \in \Sigma(\bar{A}')$ and $a' = a_0' + a_1'$, $a_i' \in A_i'$. Then

$$|\langle a', a\rangle| \leqslant |\langle a_0', a\rangle| + |\langle a_1', a\rangle| \leqslant (\|a_0'\|_{A_0'} + \|a_1'\|_{A_1'}) \max(\|a\|_{A_0}, \|a\|_{A_1}), \quad a \in \Delta(\bar{A}).$$

Consequently, $a' \in \Delta(\bar{A})'$ and $\|a'\|_{\Delta(\bar{A})'} \leqslant \|a'\|_{\Sigma(\bar{A}')}$.

Conversely, let $l \in \Delta(\bar{A})'$, i.e.,

$$|l(a)| \leqslant \|l\|_{\Delta(\bar{A})'} \|a\|_{\Delta(\bar{A})}, \quad a \in \Delta(\bar{A}).$$

Then the linear form

$$\lambda: (a_0, a_1) \mapsto l\ ((a_0 + a_1)/2)$$

on $E = \{(a_0, a_1) \in A_0 \oplus A_1 : a_0 = a_1\}$ is continuous in the norm $\max(\|a_0\|_{A_0}, \|a_1\|_{A_1})$ on $A_0 \oplus A_1$, E is a subspace of $A_0 \oplus A_1$. Then, by the Hahn-Banach theorem, there is $(a_0', a_1') \in A_0' \oplus A_1'$ such that

$$\|a_0'\|_{A_0'} + \|a_1'\|_{A_1'} \leqslant \|l\|_{\Delta(\bar{A})'}$$

and

$$\lambda(a_0, a_1) = \langle a_0', a_0 \rangle + \langle a_1', a_1 \rangle, \quad (a_0, a_1) \in E.$$

Thus, taking $a_0 = a_1 = a$, we obtain

$$l(a) = \langle a_0', a \rangle + \langle a_1', a \rangle = \langle a_0' + a_1', a \rangle, \quad a \in \Delta(\bar{A}).$$

By the density assumption, a_0' and a_1' are determined by their values on $\Delta(\bar{A})$. Putting $l = a_0' + a_1'$, $\|l\|_{\Sigma(\bar{A}')} \leqslant \|l\|_{\Delta(\bar{A})'}$ follows.

This completes the proof of the first formula. □

2.8. Exercises

1. Prove Lemma 2.3.1 in detail. In particular, use the Hausdorff property of $\mathfrak{A}$ to show that $A_0 \cap A_1$ is complete if A_0 and A_1 are complete.

2. Use Lemma 2.2.1 to prove that the space of all bounded linear operators from a normed linear space to a Banach space is complete.

3. Let X_i, $i = 0, 1$, and X be Banach spaces, X_i closed in X and $X_i \subset X$, $i = 0, 1$. Show that the following two conditions are equivalent:
(i) $X_0 + X_1$ is closed in X;
(ii) $\|x\|_{X_0 + X_1} \leqslant C \|x\|_X$ for $x \in X_0 + X_1$.

4. (Aronszajn and Gagliardo [1]). Let A be an interpolation space of exponent θ with respect to $\bar{A}$. Prove that there is a minimal interpolation functor F_θ, which is exact and of exponent θ, such that $F_\theta(\bar{A}) = A$.

Hint: Use the functional $N_\theta(x) = \sum_j \|T_j\|_{A_0, X_0}^{1-\theta} \|T_j\|_{A_1, X_1}^{\theta} \|a_j\|_A$.

5. (Aronszajn-Gagliardo [1]). Consider the category $\mathscr{B}$ of all Banach spaces and let A be an interpolation space with respect to the couple $\bar{A}$. Prove that there exists a maximal exact interpolation functor F_1 on $\mathscr{B}$, such that $F_1(\bar{A})=A$.

Hint: Define $X=F_1(\bar{A})$ as the space of all $x\in\Sigma(\bar{X})$ such that $Tx\in A$ for all $T:\bar{X}\to\bar{A}$. The norm on X is $M(x)=\sup\{\|Tx\|_A : \max(\|T\|_{X_0,A_0}, \|T\|_{X_1,A_1})\leqslant 1\}$.

6. (Gustavsson [1]). Let A_i, $i=0,1$, be seminormed linear spaces, i.e., the norms are now only semidefinite. Moreover, let $A_i\subset\mathfrak{A}$, $i=0,1$, $\mathfrak{A}$ being a linear space. Put $\bar{A}=\{A_0,A_1\}$ and

$$N(\bar{A})=\{a\in A_0+A_1 \mid \inf_{a=a_0+a_1}(\|a_0\|_{A_0}+\|a_1\|_{A_1})=0\},$$

the null space of the couple $\bar{A}$. Show that $N(\bar{A})$ is a closed linear subspace of A_0+A_1 equipped with the seminorm in the definition of $N(\bar{A})$. If $A_0\cap A_1$ is complete in the seminorm $\max(\|\cdot\|_{A_0}, \|\cdot\|_{A_1})$ and $a\in N(\bar{A})$ then prove that there exist $a_i\in A_i$ with $a_0+a_1=a$ and $\|a_i\|_{A_i}=0$, $i=0,1$.

7. (Gagliardo [2]). Let A and B be (semi-)normed linear spaces and $A\subset B$. The Gagliardo completion of A relative to B, written $A^{B,c}$, is the set of all $b\in B$ for which there exists a sequence (a_n) bounded in A and with the limit b in B.

(a) Show that $A^{B,c}$ with

$$\|b\|_{A^{B,c}}=\inf_{(a_n)}\sup_n\|a_n\|_A$$

is a (semi-)normed linear space, and that $\|b\|_{A^{B,c}}\leqslant\|b\|_A$ for $b\in A$.

(b) Show that $A^{B,c}$ is an exact interpolation space with respect to (A,B).

(c) Show that if A and B are Banach spaces, such that A is dense in B and A is reflexive, then $A^{B,c}=A$.

8. Let A and B be as in the previous exercise. The Cauchy completion of A relative to B, written A^c, is the set of all $b\in B$ for which there exists a sequence (a_n), Cauchy in A and with the limit b in B. Prove that A^c is a semi-normed linear space with

$$\|b\|_{A^c}=\inf_{(a_n)}\sup_n\|a_n\|_A,$$

and that $\|b\|_{A^c}\leqslant\|b\|_A$ for $b\in A$.

(As the notation indicates, A^c may be constructed without reference to a set B. Cf. Dunford-Schwartz [1].)

9. Show that $A^c\subset A^{B,c}$ with A^c and $A^{B,c}$ as in Exercise 7 and Exercise 8.

10. Prove the following dual corollaries to Theorem 2.6.1:

(a) If a and b satisfy

$$\begin{cases} J(t,b) \leqslant J(t,a) & \text{all} \quad t>0 \\ K(t,a) = \min(\|a\|_0, t\|a\|_1) & \text{all} \quad t>0 \end{cases}$$

then (1) implies $\|b\|_B \leqslant \|a\|_A$.

(b) If a and b satisfy

$$\begin{cases} K(t,b) \leqslant K(t,a) & \text{all} \quad t>0 \\ K(t,b) = \min(\|b\|_0, t\|b\|_1) & \text{all} \quad t>0 \end{cases}$$

then (1) implies $\|b\|_B \leqslant \|a\|_A$.

11. (Weak reiteration theorem). Let $\bar{X} = (X_0, X_1)$ and $\bar{A} = (A_0, A_1)$ be given couples.

(a) Suppose that X_0 and X_1 are (exact) interpolation spaces with respect to $\bar{A}$ and let X be an (exact) interpolation space with respect to $\bar{X}$. Then X is an (exact) interpolation space with respect to $\bar{A}$.

(b) Suppose that X_0 and X_1 are (exact) interpolation spaces of exponents θ_0 and θ_1 respectively with respect to $\bar{A}$ and that X is an (exact) interpolation space of exponent η with respect to $\bar{X}$. Then X is an (exact) interpolation space of exponent θ with respect to $\bar{A}$ provided that

$$\theta = (1-\eta)\theta_0 + \eta\theta_1.$$

12. Show that if A_0 is contained in A_1 as a set, and $\bar{A}$ is a compatible couple in $\mathcal{N}$, then

$$\|a\|_{A_0} \sim \|a\|_{\Delta(\bar{A})} \qquad (a \in A_0).$$

13. (Aronszajn-Gagliardo [1]). Let A_0 and A_1 be normed linear subspaces of a linear space $\mathfrak{A}$. Consider their direct sum $A_0 \oplus A_1$ and the set $Z \subset A_0 \oplus A_1$,

$$Z = \{(a_0, a_1) \in A_0 \oplus A_1 \mid a_0 + a_1 = 0\}.$$

Let $\|a_0\|_{A_0} + \|a_1\|_{A_1}$ be the norm on $A_0 \oplus A_1$. Show that $(A_0 \oplus A_1)/Z$, with the quotient norm, is isometrically isomorphic to $A_0 + A_1$ and that the same is true for Z, with norm $\max(\|a_0\|_{A_0}, \|a_1\|_{A_1})$, and $A_0 \cap A_1$. (The definitions of $A_0 + A_1$ and $A_0 \cap A_1$ and their respective norms are found in Section 2.4.)

14. (Girardeau [1]). (a) Let A_i $(i=0,1)$ be locally convex Hausdorff topological vector spaces, such that A_0 is subspace of A_1. Assume that there is an antilinear surjective mapping $M: A_1' \to A_0$ satisfying

$$\langle Ma, a \rangle \geqslant 0 \qquad (a \in A_1').$$

Show that M defines a scalar product

$$(c,d)=\langle Ma,b\rangle,$$

where $c=Ma$ and $d=Mb$, and that if A_1 is quasicomplete then the completion of A_0 in the scalar product topology is a Hilbert space A with

$$A_0\subset A\subset A_1.$$

(b) Let A_i and B_i be as in (a), and assume that A_0 and B_0 are dense in A and B respectively. Consider a continuous and linear mapping $T\colon A_1\to B_1$, with $T(A_0)$ contained in B_0. Show that

$$T\in L(A,B)$$

iff there exists a $\lambda>0$, such that

$$\textstyle\sum_{n=0}^{\infty}(\lambda^{-1}T^*T)^n(a) \qquad (a\in A_0)$$

converges weakly in the completion of A_1.

Hint: $\langle (T^*T)^n a,b\rangle\geqslant 0$ if $Mb=a$.

2.9. Notes and Comment

The origin of the study of interpolation spaces was, as we noted in Chapter 1, interpolation with respect to couples of L_p-spaces. Interpolation with respect to more general couples, i.e., Hilbert couples, Banach couples, etc., seems to have been introduced in the late fifties. Several interpolation methods have been invented. A few of the relevant, but not necessarily the first, references are: Lions [1], "espaces de trace"; Krein [1], "normal scales of spaces"; Gagliardo [2], "unified structure"; Lions and Peetre [1], "classe d'espaces d'interpolation", Calderón [2], "the complex method". We shall discuss their relation in Chapters 3—5. Two of these interpolation methods will be treated in some detail: the real method, which is essentially that of Lions and Peetre [1], and the complex method. This is done in the following two chapters.

For interpolation results pertaining to *couples of locally convex topological spaces*, see e.g. Girardeau [1] (cf. Exercise 14) and Deutsch [1]. Interpolation with respect to *couples of quasinormed Abelian groups* has been treated by Peetre and Sparr [1] (see Section 3.10 and Chapter 7).

Non-linear interpolation has been considered, e.g., by Gagliardo [1], Peetre [17], Tartar [1], Brézis [1]. For additional references, see Peetre [17]. (Cf. also Gustavsson [2].) "Non-linear" indicates that non-linear operators are admitted: e.g., *Lipschitz and Hölder operators*. Cf. Section 3.13. There are applications to partial differential equations: Tartar [1], Brézis [1]. See also Section 7.6.

2.9.1—2. The functorial approach to interpolation merely provides a convenient framework for the underlying primitive ideas, and makes the exposition more stringent.

2.9.3. Introducing couples $\bar{A}=(A_0, A_1)$ we have assumed the existence of a Hausdorff topological vector space $\mathfrak{A}$, such that $A_i \subset \mathfrak{A}$ $(i=0,1)$. This assumption is made for convenience only. Cf. Aronszajn and Gagliardo [1], where $A_0 \oplus A_1$ plays the role of $\mathfrak{A}$, but, anyway, they have to make additional assumptions in order to obtain unique limits. This property, and the possibility of forming A_0+A_1, are the essential consequences of the requirements $A_i \subset \mathfrak{A}$ $(i=0,1)$, $\mathfrak{A}$ Hausdorff. (Cf. Exercise 1 and 13.)

Peetre [20] has coined the notion *weak couple* for the situation when A_0 and A_1 only have continuous and linear injections into a Hausdorff topological vector space $\mathfrak{A}$ (cf. also Gagliardo [1]). $\Sigma(\bar{A})$ may then still be viewed as a subspace of $\mathfrak{A}$: the linear hull of $i_0(A_0)$ and $i_1(A_1)$, but $\Delta(\bar{A})$ is the subspace of $A_0 \oplus A_1$ of those (a_0, a_1) for which $i_0(a_0)=i_1(a_1)$.

2.9.4—5. Concerning the relation between the concepts "interpolation space with respect to $\bar{A}$" and "interpolation spaces with respect to $\bar{A}$ and $\bar{B}$", see Aronszajn and Gagliardo [1]. They show, however, that if A' is maximal and B' is minimal among all spaces satisfying (3), then A' is an interpolation space with respect to $\bar{A}$ and likewise for B' and $\bar{B}$.

The definition of "interpolation space" implies the uniform interpolation condition (5) if the spaces labelled by the letter B are Banach spaces (i.e., the spaces labelled by the letter A need not be complete in Theorem 2.4.2). On the other hand, we do not know of any interpolation space that is not uniform. This question is connected with the Aronszajn-Gagliardo theorem, since Theorem 2.4.2 is used in its proof. Thus there is a question whether the Aronszajn-Gagliardo theorem holds also in some category larger than $\mathscr{B}$, say $\mathscr{N}$. Obviously, our proof breaks down, because we invoke the Banach theorem, a consequence of Baire's category theorem, and in these theorems completeness is essential.

2.9.6. The necessary condition is valid also in the semi-normed case (cf. Exercise 6). This necessary condition, adapted to a specific couple and more or less disguised, has been used by several authors to determine whether or not a certain space may be an interpolation space with respect to a given couple. (Cf. Bergh [1] and 5.8.)

2.9.7. The duality theorem is taken over from Lions and Peetre [1].

2.9.8. Using the Gagliardo completion, Exercise 7, Aronszajn and Gagliardo [1] have shown that, in the category $\mathscr{B}$ and in general, A_0 and A_1 are not interpolation spaces with respect to the (compatible) couple $(\Delta(\bar{A}), \Sigma(\bar{A}))$. This fact should be viewed in contrast to the statement that $\Delta(\bar{A})$ and $\Sigma(\bar{A})$ always are interpolation spaces with respect to the couple $\bar{A}$ (see Section 2.4 and also compare Section 5.8).

Chapter 3

The Real Interpolation Method

In this chapter we introduce the first of the two explicit interpolation functors which we employ for the applications in the last three chapters. Our presentation of this method/functor—the real interpolation method—follows essentially Peetre [10]. In general, we work with normed linear spaces. However, we have tried to facilitate the extension of the method to comprise also the case of quasi-normed linear spaces, and even quasi-normed Abelian groups. Consequently, these latter cases are treated with a minimum of new proofs in Sections 3.10 and 3.11. In the first nine sections we consider the category $\mathscr{N}_1$ of compatible couples of spaces in the category $\mathscr{N}$ of normed linear spaces unless otherwise stated.

3.1. The K-Method

In this section we consider the category $\mathscr{N}$ of all normed vector spaces. We shall construct a family of interpolation functors $K_{\theta,p}$ on the category $\mathscr{N}$.

We know that Σ is an interpolation functor on $\mathscr{N}$. The norm on $\Sigma(\bar{A})$ is

$$\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+\|a_1\|_{A_1}),$$

if $\bar{A}=(A_0,A_1)$. Now we can replace the norm on A_1 by an equivalent one. We may, for instance, replace the norm $\|a_1\|_{A_1}$ by $t\cdot\|a_1\|_{A_1}$, where t is a fixed positive number. This means that

$$K(t,a)=K(t,a;\bar{A})=\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+t\,\|a_1\|_{A_1})$$

is an equivalent norm on $\Sigma(\bar{A})$ for every fixed $t>0$. More precisely, we have the following lemma.

Lemma 3.1.1. *For any* $a\in\Sigma(\bar{A}), K(t,a)$ *is a positive, increasing and concave function of* t. *In particular*

(1) $\qquad K(t,a)\leqslant \max(1,t/s)\,K(s,a)$. $\square$

The lemma is a direct consequence of the definition, and is left as an exercise for the reader. Moreover, (1) implies at once that $K(t,a)$ is an equivalent norm on $\Sigma(\bar{A})$ for each fixed positive t.

The functional $t \to K(t,a)$, $a \in \Sigma(\bar{A})$, has a geometrical interpretation in the *Gagliardo diagram.* Consider the set $\Gamma(a)$,

$$\Gamma(a) = \{x = (x_0, x_1) \in \mathbb{R}^2 \mid \exists\, a_0 + a_1 = a,\ a_i \in A_i,\ i = 0, 1;\ \|a_i\|_{A_i} \leqslant x_i\}.$$

It is immediately verified that $\Gamma(a)$ is a convex subset of $\mathbb{R}^2$, cf. Figure 3. In addition

(2) $$K(t,a) = \inf_{x \in \Gamma(a)}(x_0 + t x_1) = \inf_{x \in \partial\Gamma(a)}(x_0 + t x_1),$$

i.e. $K(t,a)$ is the x_0-intercept of the tangent to $\partial\Gamma(a)$ (boundary of $\Gamma(a)$), with slope $-t^{-1}$. This follows from the fact that $K(t,a)$ is a positive, increasing and concave function and thus also continuous.

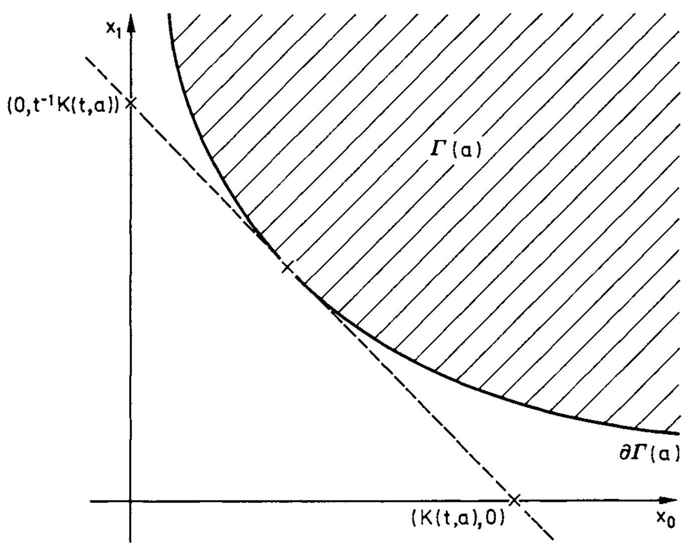


Fig. 3

For every $t > 0$, $K(t,a)$ is a norm on the interpolation space $\Sigma(\bar{A})$. We now define a new interpolation space by means of a kind of superposition, which is obtained by imposing conditions on the function $t \to K(t,a)$. Let $\Phi_{\theta,q}$ be the functional defined by

(3) $$\Phi_{\theta,q}(\varphi(t)) = \left(\int_0^\infty (t^{-\theta}\varphi(t))^q\, dt/t\right)^{1/q}, \quad 1 \leqslant q \leqslant \infty,$$

where φ is a non-negative function. Then we consider the condition

(4) $$\Phi_{\theta,q}(K(t,a)) < \infty.$$

By Lemma 3.1.1 we see that this condition is meaningful in the cases $0<\theta<1, 1\leqslant q\leqslant\infty$ and $0\leqslant\theta\leqslant 1$, $q=\infty$. *For these values of* θ *and* q, *we let* $\bar{A}_{\theta,q;K}=K_{\theta,q}(\bar{A})$ *denote the space of all* $a\in\Sigma(\bar{A})$, *such that* (4) *holds.* We put

(5) $$\|a\|_{\theta,q;K}=\Phi_{\theta,q}(K(t,a)).$$

In the following theorem it is understood that if $T:\bar{A}\to\bar{B}$ then $K_{\theta,q}(T)=T$.

3.1.2. Theorem. $K_{\theta,q}$ *is an exact interpolation functor of exponent* θ *on the category* $\mathscr{N}$. *Moreover, we have*

(6) $$K(s,a;\bar{A})\leqslant\gamma_{\theta,q}s^{\theta}\|a\|_{\theta,q;K}.$$

Proof: Since $K(t,a;\bar{A})$ is a norm on $\Sigma(\bar{A})$, and since $\Phi_{\theta,q}$ has all the three properties of a norm, it is easy to see that $K_{\theta,q}(\bar{A})$ is a normed vector space.

In order to prove (6), we use Formula (1) of Lemma 3.1.1 which can be written in the form

$$\min(1,t/s)\,K(s,a)\leqslant K(t,a).$$

Applying $\Phi_{\theta,q}$ to this inequality we get

$$\Phi_{\theta,q}(\min(1,t/s))\,K(s,a)\leqslant\|a\|_{\theta,q;K}.$$

Now we note that, with $s>0$,

$$\begin{aligned}\Phi_{\theta,q}(\varphi(t/s))&=(\textstyle\int_0^\infty(t^{-\theta}\varphi(t/s))^q\,dt/t)^{1/q}\\&=s^{-\theta}(\textstyle\int_0^\infty((t/s)^{-\theta}\varphi(t/s))^q\,d(t/s)/(t/s))^{1/q},\end{aligned}$$

i.e.

(7) $$\Phi_{\theta,q}(\varphi(t/s))=s^{-\theta}\Phi_{\theta,q}(\varphi(t)).$$

Thus

$$\Phi_{\theta,q}(\min(1,t/s))=s^{-\theta}\Phi_{\theta,q}(\min(1,t)).$$

Since $\Phi_{\theta,q}(\min(1,t))=1/(q^{1/q}(\theta(1-\theta))^{1/q})$, we obtain (6).

Using (6) with $s=1$, we see that $K_{\theta,q}(\bar{A})\subset\Sigma(\bar{A})$. The inclusion $\Delta(\bar{A})\subset K_{\theta,q}(\bar{A})$ is obvious, since

$$K(t,a)\leqslant\min(1,t)\,\|a\|_{\Delta(\bar{A})}.$$

In fact, this inequality gives

$$\|a\|_{\theta,q;K}\leqslant\Phi_{\theta,q}(\min(1,t))\,\|a\|_{\Delta(\bar{A})}.$$

It remains to prove that $K_{\theta,q}$ is an exact interpolation functor of exponent θ. Thus, suppose that $T: \bar{A}\to\bar{B}$, where $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$. Put

$$M_j=\|T\|_{A_j,B_j},\quad j=0,1.$$

Then

$$K(t,\,Ta;\bar{B})\leqslant \inf_{a=a_0+a_1}(\|Ta_0\|_{B_0}+t\,\|Ta_1\|_{B_1})$$
$$\leqslant \inf_{a=a_0+a_1}(M_0\,\|a_0\|_{A_0}+tM_1\,\|a_1\|_{A_1}).$$

Thus

$$K(t,\,Ta;\bar{B})\leqslant M_0\,K(M_1t/M_0,a;\bar{A}). \tag{8}$$

But, using (7) with $s=M_0/M_1$, we obtain

$$\|Ta\|_{K_{\theta,q}(\bar{B})}\leqslant M_0^{1-\theta}M_1^{\theta}\,\|a\|_{K_{\theta,q}(\bar{A})}.$$

This proves that $K_{\theta,q}$ is an exact interpolation functor of exponent θ. □

Remark: The interpolation property holds for all operators $T:\Sigma(\bar{A})\to\Sigma(\bar{B})$, such that (8) holds. In particular, the interpolation property holds for all operators T such that $T(a_0+a_1)=b_0+b_1$ where $\|b_j\|_{B_j}\leqslant M_j\,\|a_j\|_{A_j}$, $j=0,1$.

There are several useful variants of the $K_{\theta,q}$-functor. In this section we shall mention only the discrete $K_{\theta,q}$-method. We shall replace the continuous variable t by a discrete variable v. The connection between t and v is $t=2^v$. This discretization will turn out to be a most useful technical device.

Let us *denote by* $\lambda^{\theta,q}$ *the space of all sequences* $(\alpha_v)_{-\infty}^{\infty}$, *such that*

$$\|(\alpha_v)\|_{\lambda^{\theta,q}}=\left(\textstyle\sum_{-\infty}^{\infty}(2^{-v\theta}|\alpha_v|)^q\right)^{1/q}<\infty\,.$$

3.1.3. Lemma. *If* $a\in\Sigma(\bar{A})$ *we put* $\alpha_v=K(2^v,a;\bar{A})$. *Then* $a\in K_{\theta,q}(\bar{A})$ *if and only if* $(\alpha_v)_{-\infty}^{\infty}$ *belongs to* $\lambda^{\theta,q}$. *Moreover, we have*

$$2^{-\theta}\log 2\,\|\alpha_v\|_{\lambda^{\theta,q}}\leqslant\|a\|_{\theta,q;K}\leqslant 2\cdot\log 2\,\|\alpha_v\|_{\lambda^{\theta,q}}.$$

Proof: Clearly, we have

$$\|a\|_{\theta,q;K}=\left(\textstyle\sum_{-\infty}^{\infty}\int_{2^v}^{2^{v+1}}(t^{-\theta}K(t,a))^q\,dt/t\right)^{1/q}.$$

Now Lemma 3.1.1 implies that

$$K(2^v,a)\leqslant K(t,a)\leqslant 2\,K(2^v,a),\quad 2^v\leqslant t\leqslant 2^{v+1}.$$

Consequently,

$$2^{-\theta}2^{-\nu\theta}\alpha_\nu \leqslant t^{-\theta}K(t,a) \leqslant 2\cdot 2^{-\nu\theta}\alpha_\nu, \quad 2^\nu \leqslant t \leqslant 2^{\nu+1},$$

and thus the inequalities of the lemma follow. □

3.2. The J-Method

There is a definition of the J-method which is similar to the description of the K-method in the previous section. Instead of starting with the interpolation method Σ we start with the functor Δ and define the J-method by means of a kind of superposition.

For any fixed $t>0$ we put

$$J(t,a) = J(t,a;\bar{A}) = \max(\|a\|_{A_0}, t\,\|a\|_{A_1}),$$

for $a \in \Delta(\bar{A})$. Clearly $J(t,a)$ is an equivalent norm on $\Delta(\bar{A})$ for a given $t>0$. More precisely we have the following lemma, the proof of which is immediate, and is left as an exercise for the reader.

3.2.1. Lemma. *For any $a \in \Delta(\bar{A})$, $J(t,a)$ is a positive, increasing and convex function of t, such that*

(1) $$J(t,a) \leqslant \max(1, t/s)\, J(s,a),$$

(2) $$K(t,a) \leqslant \min(1, t/s)\, J(s,a). \quad □$$

The space $\bar{A}_{\theta,q;J} = J_{\theta,q}(\bar{A})$ is now defined as follows. *The elements a in $J_{\theta,q}(\bar{A})$ are those in $\Sigma(\bar{A})$ which can be represented by*

(3) $$a = \int_0^\infty u(t)\,dt/t \quad (\textit{convergence in } \Sigma(\bar{A})),$$

where $u(t)$ is measurable with values in $\Delta(\bar{A})$ and

(4) $$\Phi_{\theta,q}(J(t,u(t))) < \infty.$$

Here we consider the cases $0<\theta<1$, $1 \leqslant q \leqslant \infty$ and $0 \leqslant \theta \leqslant 1$, $q=1$. We put

(5) $$\|a\|_{\theta,q;J} = \inf_u \Phi_{\theta,q}(J(t,u(t))),$$

where the infimum is taken over all u such that (3) *and* (4) *hold.*

3.2.2. Theorem. *Let $J_{\theta,q}$ be defined by* (3), (4) *and* (5). *Then $J_{\theta,q}$ is an exact interpolation functor of exponent θ on the category $\mathcal{N}$. Moreover, we have*

$$\text{(6)} \qquad \|a\|_{\theta,q;J} \leqslant C s^{-\theta} J(s, a; \bar{A}), \qquad a \in \Delta(\bar{A})$$

where C is independent of θ and q.

Proof: Obviously, $\|a\|_{\theta,q;J}$ is a norm. Assume that $T: A_j \to B_j$, with norm M_j, $j=0,1$. For $a \in \bar{A}_{\theta,q;J}$, we have, since $T: \Sigma(\bar{A}) \to \Sigma(\bar{B})$ is bounded linear, that $Tu(t)$ is measurable,

$$Ta = T\left(\int_0^\infty u(t)\,dt/t\right) = \int_0^\infty Tu(t)\,dt/t \quad (\textit{convergence in } \Sigma(\bar{B})).$$

Thus, with this u,

$$\begin{aligned} J(t, Tu(t)) &= \max(\|Tu(t)\|_{B_0}, t\,\|Tu(t)\|_{B_1} \\ &\leqslant M_0 \max(\|u(t)\|_{A_0}, tM_1/M_0\,\|u/t)\|_{A_1}) \\ &= M_0 J(tM_1/M_0, u(t)), \end{aligned}$$

and we obtain, by the properties of $\Phi_{\theta q}$,

$$\Phi_{\theta,q}(J(t, Tu(t))) \leqslant M_0^{1-\theta} M_1^{\theta}\, \Phi_{\theta q}(J(t, u(t))).$$

Taking the infimum of the right hand side, we infer that $J_{\theta q}$ is an exact interpolation functor. Finally, noting that $a \in \Delta(\bar{A})$ has the representation

$$a = (\log 2)^{-1} \int_1^2 a\,dt/t = (\log 2)^{-1} \int_0^\infty a \cdot \chi_{(1,2)}(t)\,dt/t,$$

(6) follows at once from (1). $\square$

There is a discrete representation of the space $J_{\theta,q}(\bar{A})$, which is analogous to the discrete representation of the space $K_{\theta,q}(\bar{A})$.

3.2.3. Lemma. $a \in J_{\theta q}(\bar{A})$ *iff there exist* $u_\nu \in \Delta(\bar{A})$, $-\infty < \nu < \infty$, *with*

$$\text{(7)} \qquad a = \sum_\nu u_\nu \qquad (\textit{convergence in } \Sigma(\bar{A})),$$

and such that $(J(2^\nu, u_\nu)) \in \lambda^{\theta,q}$. *Moreover*

$$\|a\|_{\theta,q;J} \sim \inf_{(u_\nu)} \|(J(2^\nu, u_\nu))\|_{\lambda^{\theta,q}},$$

where the infimum is extended over all sequences (u_ν) *satisfying* (7).

Proof: Suppose that $a \in J_{\theta q}(\bar{A})$. Then we have a representation $a = \int_0^\infty u(t)\,dt/t$. Choose $u_\nu = \int_{2^\nu}^{2^{\nu+1}} u(t)\,dt/t$. Clearly (7) holds with these u_ν. In addition, by (1), we obtain

$$\|(J(2^\nu, u_\nu))\|_{\lambda^{\theta,q}}^q = \sum_\nu (2^{-\nu\theta} J(2^\nu, u_\nu))^q$$
$$\leqslant \sum_\nu C \int_{2^\nu}^{2^{\nu+1}} (t^{-\theta} J(t, u(t)))^q\,dt/t = C\{\Phi_{\theta q}(J(t, u(t)))\}^q,$$

and thus, taking the infimum, we conclude that

$$\inf_{(u_\nu)} \|(J(2^\nu, u_\nu))\|_{\lambda^{\theta,q}} \leqslant C\,\|a\|_{\theta,q;J}.$$

Conversely, assume that $a = \sum_\nu u_\nu$ and $(J(2^\nu, u_\nu))_\nu \in \lambda^{\theta,q}$. Choose $u(t) = u_\nu/\log 2$, $2^\nu \leqslant t < 2^{\nu+1}$. Then we obtain

$$a = \sum_\nu u_\nu = \sum_\nu \int_{2^\nu}^{2^{\nu+1}} (u_\nu/\log 2)\,dt/t = \int_0^\infty u(t)\,dt/t.$$

Also, by (1), we have

$$\{\Phi_{\theta q}(J(t, u(t)))\}^q = \int_0^\infty (t^{-\theta} J(t, u(t)))^q\,dt/t$$
$$= \sum_\nu \int_{2^\nu}^{2^{\nu+1}} (t^{-\theta} J(t, u(t)))^q\,dt/t$$
$$\leqslant \sum_\nu C(2^{-\nu\theta} J(2^\nu, u_\nu))^q.$$

Again, taking infimum, we obtain

$$\|a\|_{\theta q;J} \leqslant C \inf_{(u_\nu)} \|(J(2^\nu, u_\nu))\|_{\lambda^{\theta,q}}. \quad \square$$

3.3. The Equivalence Theorem

In this section we shall prove that the K- and J-methods of the preceding two sections are equivalent. More precisely, we shall prove the following result.

3.3.1. Theorem (The equivalence theorem). *If $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$ then $J_{\theta,q}(\bar{A}) = K_{\theta,q}(\bar{A})$ with equivalence of norms.*

Proof: Take first $a \in J_{\theta,q}(\bar{A})$ and $a = \int_0^\infty u(t)\,dt/t$. Then, by Lemma 3.2.1, it follows that

$$K(t, a) \leqslant \int_0^\infty K(t, u(s))\,ds/s \leqslant \int_0^\infty \min(1, t/s)\, J(s, u(s))\,ds/s$$
$$= \int_0^\infty \min(1, s^{-1})\, J(ts, u(ts))\,ds/s.$$

Applying $\Phi_{\theta,q}$ and changing variable, we obtain

$$\|a\|_{\theta,q;K} \leqslant \Phi_{\theta,q}(J(t, u(t))) \int_0^\infty s^\theta \min(1, s^{-1})\,ds/s = C\,\Phi_{\theta,q}(J(t, u(t))).$$

Thus $\|a\|_{\theta,q;K} \leqslant C\|a\|_{\theta,q;J}$ follows by taking the infimum.
For the converse inequality, we need a lemma.

3.3.2. Lemma (The fundamental lemma of interpolation theory). *Assume that*

$$\min(1, 1/t)\,K(t,a) \to 0 \quad \textit{as} \quad t \to 0 \quad \textit{or as} \quad t \to \infty .$$

Then, for any $\varepsilon > 0$, *there is a representation*

$$a = \sum\nolimits_\nu u_\nu \qquad (\textit{convergence in } \Sigma(\bar{A}))$$

of a, such that

$$J(2^\nu, u_\nu) \leqslant (\gamma + \varepsilon)\,K(2^\nu, a).$$

Here γ *is a universal constant* $\leqslant 3$.

Before we prove the fundamental lemma, we complete the proof of Theorem 3.3.1. By Theorem 3.1.3, we have

$$K(t,a) \leqslant C_{\theta,q} t^\theta \|a\|_{\theta,q;K}$$

for any given $a \in K_{\theta,q}(\bar{A})$. Thus it follows that $\min(1, 1/t)K(t,a) \to 0$ as $t \to 0$ or $t \to \infty$. Consequently, the fundamental lemma implies the existence of a representation $a = \sum_\nu u_\nu$, such that

$$J(2^\nu, u_\nu) \leqslant (\gamma + \varepsilon)\,K(2^\nu, a).$$

Thus

$$\|(J(2^\nu, u_\nu))\|_{\lambda^{\theta,q}} \leqslant (\gamma + \varepsilon)\,\|(K(2^\nu, a))\|_{\lambda^{\theta,q}} .$$

By Lemma 3.1.3 and 3.2.2, we see

$$\|a\|_{\theta,q;J} \leqslant 4(\gamma + \varepsilon)\,\|a\|_{\theta,q;K} .$$

This completes the proof of Theorem 3.3.1. □

Proof of the fundamental lemma: For every integer ν, there is a decomposition $a = a_{0,\nu} + a_{1,\nu}$, such that for given $\varepsilon > 0$

$$(1) \qquad \|a_{0,\nu}\|_{A_0} + 2^\nu \|a_{1,\nu}\|_{A_1} \leqslant (1+\varepsilon)K(2^\nu, a).$$

Thus it follows that

$$\|a_{0,\nu}\|_{A_0} \to 0 \quad \text{as} \quad \nu \to -\infty,$$

$$\|a_{1,\nu}\|_{A_1} \to 0 \quad \text{as} \quad \nu \to +\infty .$$

Write

$$u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu}.$$

Then $u_\nu \in \Delta(\bar{A})$ and

$$a - \sum_{-N}^{M} u_\nu = a - a_{0,M} + a_{0,-N-1} = a_{0,-N-1} + a_{1,M}.$$

Therefore, we have

$$K(1, a - \sum_{-N}^{M} u_\nu) \leqslant \|a_{0,-N-1}\|_{A_0} + \|a_{1,M}\|_{A_1}.$$

Letting $N \to \infty$ and $M \to \infty$, we see that

$$a = \sum_{-\infty}^{\infty} u_\nu \qquad (\textit{convergence in } \Sigma(\bar{A})).$$

By (1), we also see that

$$\begin{aligned} J(2^\nu, u_\nu) &\leqslant \max(\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0}, 2^\nu(\|a_{1,\nu-1}\|_{A_1} + \|a_{1,\nu}\|_{A_1})) \\ &\leqslant 3(1+\varepsilon) K(2^\nu, a). \end{aligned}$$

This proves the lemma. □

In the sequel we shall speak of the real interpolation method. Then we shall mean either the $K_{\theta,q}$- or the $J_{\theta,q}$-method. In view of the equivalence theorem, these two methods give the same result if $0<\theta<1$. Accordingly, we shall write $\bar{A}_{\theta,q}$ instead of $\bar{A}_{\theta,q;K}$ or $\bar{A}_{\theta,q;J}$ if $0<\theta<1$. If $\theta=0$ or 1 and $q=\infty$, we shall let $\bar{A}_{\theta,q}$ denote the space $\bar{A}_{\theta,q;K}$. The norm on $\bar{A}_{\theta,q}$ we denote by $\|\cdot\|_{\theta,q}$ if $0<\theta<1$ or if $0 \leqslant \theta \leqslant 1$ and $q=\infty$.

3.4. Simple Properties of $\bar{A}_{\theta,q}$

In this section we shall prove some basic and simple properties of $\bar{A}_{\theta,q}$. We collect these results in two theorems, the first of which deals with inclusions between various $\bar{A}_{\theta,q}$-spaces.

3.4.1. Theorem. *Let $\bar{A}=(A_0, A_1)$ be a given couple. Then we have*

(a) $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$ *(with equal norms);*

(b) $\bar{A}_{\theta,q} \subset \bar{A}_{\theta,r}$ *if* $q \leqslant r$;

(c) $\bar{A}_{\theta_0,q_0} \cap \bar{A}_{\theta_1,q_1} \subset \bar{A}_{\theta,q}$ *if* $\theta_0 < \theta < \theta_1$;

(d) $A_1 \subset A_0 \Rightarrow \bar{A}_{\theta_1,q} \subset \bar{A}_{\theta_0,q}$ *if* $\theta_0 < \theta_1$;

(e) $A_1 = A_0$ *(equal norms) implies* $\bar{A}_{\theta,q} = A_0$ *and* $\|a\|_{A_0} = (q\theta(1-\theta))^{1/q} \|a\|_{\theta,q}$.

Proof: We have

$$K(t, a; A_0, A_1) = t\,K(t^{-1}, a; A_1, A_0),$$

and $\Phi_{\theta,q}(\varphi(t)) = \Phi_{1-\theta,q}(t\varphi(t^{-1}))$. This gives (a).

In order to prove (b), we first note that Theorem 3.1.2 implies (b) when $r=\infty$. If $q \leqslant r < \infty$ we obtain, again by Theorem 3.1.2,

$$\|a\|_{\theta,r} = (\textstyle\int_0^\infty (t^{-\theta}K(t,a))^q (t^{-\theta}K(t,a))^{r-q}\,dt/t)^{1/r} \leqslant C\,\|a\|_{\theta,q}^{q/r}\,\|a\|_{\theta,q}^{1-q/r},$$

which gives (b).

For the proof of (c), we note that

$$\Phi_{\theta,q}(\varphi) \leqslant (\textstyle\int_0^1 (t^{-\theta}\varphi(t))^q\,dt/t)^{1/q} + (\int_1^\infty (t^{-\theta}\varphi(t))^q\,dt/t)^{1/q}.$$

Now it is easy to see that the first integral can be estimated by $\Phi_{\theta_1,q_1}(\varphi)$, and the second one by $\Phi_{\theta_0,q_0}(\varphi)$. This proves (c).

If $A_1 \subset A_0$ we have $\|a\|_{A_0} \leqslant k\,\|a\|_{A_1}$. Then $K(t,a;\bar{A}) = \|a\|_{A_0}$ if $t>k$. In fact, if $a = a_0 + a_1$ we have

$$\|a\|_{A_0} \leqslant \|a_0\|_{A_0} + t/k \cdot \|a_1\|_{A_0} \leqslant \|a_0\|_{A_0} + t\,\|a_1\|_{A_1}$$

which proves $\|a\|_{A_0} \leqslant K(t,a;\bar{A})$. It follows that

$$\|a\|_{\theta,q} \sim (\textstyle\int_0^k (t^{-\theta}K(t,a;\bar{A}))^q\,dt/t)^{1/q} + \|a\|_{A_0}.$$

This implies (d). Sinse (e) is immediate, the theorem follows. □

3.4.2. Theorem. *Let $\bar{A} = (A_0, A_1)$ be a given couple.*

(a) *If A_0 and A_1 are complete then so is $\bar{A}_{\theta,q}$.*

(b) *If $q<\infty$ then $\Delta(\bar{A})$ is dense in $\bar{A}_{\theta,q}$.*

(c) *The closure of $\Delta(\bar{A})$ in $\bar{A}_{\theta,\infty}$ is the space $\bar{A}^0_{\theta,\infty}$ of all a such that*

$$t^{-\theta}K(t,a;\bar{A}) \to 0 \quad \text{as} \quad t\to 0 \quad \text{or} \quad t\to\infty.$$

(d) *If A_j^0 denotes the closure of $\Delta(\bar{A})$ in A_j we have for $q<\infty$,*

$$(A_0, A_1)_{\theta,q} = (A_0^0, A_1)_{\theta,q} = (A_0, A_1^0)_{\theta,q} = (A_0^0, A_1^0)_{\theta,q}.$$

Proof: In order to prove (a), we use Lemma 2.2.1. Assume that

$$\textstyle\sum_j \|a_j\|_{\theta,q} < \infty.$$

By Theorem 3.1.2, we have $K(1,a_j) \leqslant C\,\|a_j\|_{\theta,q}$. By Lemma 2.3.1, we know that $A_0 + A_1$ is complete. Thus $\sum_j a_j$ converges in $A_0 + A_1$ to an element a. Since

$$\Phi_{\theta,q}(K(t, \textstyle\sum_{j>N} a_j)) \leqslant \Phi_{\theta,q}(\sum_{j>N} K(t,a_j)) \leqslant \sum_{j>N} \Phi_{\theta,q}(K(t,a_j)),$$

it follows that $a \in \bar{A}_{\theta,q}$ and $\sum_j a_j$ converges to a in $\bar{A}_{\theta,q}$.

We now prove (b). Note that the assumption $q<\infty$ implies $0<\theta<1$. Then every $a \in \bar{A}_{\theta,q}$ may be represented by $a=\sum_\nu u_\nu$, where $u_\nu \in \Delta(\bar{A})$ and

$$(\sum_\nu (2^{-\nu\theta} J(2^\nu, u_\nu))^q)^{1/q} < \infty .$$

Then

$$\|a - \sum_{|\nu| \leqslant N} u_\nu\|_{\theta,q} \leqslant (\sum_{|\nu|>N} (2^{-\nu\theta} J(2^\nu, u_\nu))^q)^{1/q} \to 0, \qquad N \to \infty .$$

This proves (b).

In (c) we assume $0 \leqslant \theta \leqslant 1$. If $a \in \bar{A}^0_{\theta,\infty}$ we obtain (from the fundamental lemma of interpolation theory (Lemma 3.3.2)) $a=\sum_\nu u_\nu$, where $u_\nu \in \Delta(\bar{A})$ and $J(2^\nu, u_\nu) \leqslant C\, K(2^\nu, a)$. Then

$$\|a - \sum_{|\nu| \leqslant N} u_\nu\|_{\theta,\infty} \leqslant C \sup_{|\nu| \geqslant N} 2^{-\nu\theta} K(2^\nu, a) \to 0, \qquad N \to \infty .$$

Thus $\Delta(\bar{A})$ is dense in $\bar{A}^0_{\theta,\infty}$. Conversely, if a is in the closure of $\Delta(\bar{A})$ in $\bar{A}_{\theta,\infty}$ then we can find $b \in \Delta(\bar{A})$ such that $\|a-b\|_{\theta,\infty} < \varepsilon$. By Lemma 3.2.1 and Theorem 3.1.2, we obtain $K(t,a) \leqslant K(t,a-b)+K(t,b) \leqslant C t^\theta \|a-b\|_{\theta,\infty} + \min(1,t) J(1,b)$. Thus

$$t^{-\theta} K(t,a) \leqslant C\varepsilon + t^{-\theta} \min(1,t) J(1,b) .$$

It follows that $a \in \bar{A}^0_{\theta,\infty}$.

The last parts (d) and (e) of the theorem are obvious. □

3.5. The Reiteration Theorem

According to the weak reiteration theorem (cf. 2.8.11), we know that if X_0 and X_1 are interpolation spaces of exponents θ_0 and θ_1 with respect to $\bar{A}$, and if X is an interpolation space of exponent η with respect to $\bar{X}=(X_0, X_1)$, then X is an interpolation space of exponent $\theta=(1-\eta)\theta_0+\eta\theta_1$ with respect to $\bar{A}$. In this section we shall prove that if X_0 and X_1 are constructed from the couple $\bar{A}$ by means of the real interpolation method, and if X is constructed from $\bar{X}$ by means of the real method then X can be constructed from $\bar{A}$ by means of the real method. Thus there is stability for repeated use of the real interpolation method.

3.5.1. Definition. *Let $\bar{A}$ be a given couple of normed vector spaces. Suppose that X is an intermediate space with respect to $\bar{A}$. Then we say that*

(a) *X is of class $\mathscr{C}_K(\theta;\bar{A})$ if $K(t,a;\bar{A}) \leqslant C t^\theta \|a\|_X$, $\quad a \in X$;*

(b) *X is of class $\mathscr{C}_J(\theta;\bar{A})$ if $\|a\|_X \leqslant C t^{-\theta} J(t,a;\bar{A})$, $\quad a \in \Delta(\bar{A})$.*

Here $0 \leqslant \theta \leqslant 1$. We also say that X is of class $\mathscr{C}(\theta;\bar{A})$ if X is of class $\mathscr{C}_K(\theta;\bar{A})$ and of class $\mathscr{C}_J(\theta;\bar{A})$.

From Theorem 3.1.2 and 3.2.3, we see that $\bar{A}_{\theta,q}$ is of class $\mathscr{C}(\theta;\bar{A})$ if $0<\theta<1$. If $\bar{A}=(A_0, A_1)$ we have that A_0 is of class $\mathscr{C}(0;\bar{A})$ and A_1 is of class $\mathscr{C}(1;\bar{A})$. This follows at once from the definition of $J(t, a;\bar{A})$ and from the inequality

$$K(t, a;\bar{A})\leqslant \min(\|a\|_{A_0}, t\|a\|_{A_1}).$$

It is sometimes convenient to write down the definition without explicit use of $K(t, a)$ and $J(t, a)$. Indeed, it is obvious that

(a) X is of class $\mathscr{C}_K(\theta;\bar{A})$ if and only if for any $t>0$ there exist $a_0\in A_0$ and $a_1\in A_1$, such that $a=a_0+a_1$ and $\|a_0\|_{A_0}\leqslant Ct^{\theta}\|a\|_X$ and $\|a_1\|_{A_1}\leqslant Ct^{\theta-1}\|a\|_X$.

We can also show that

(b) X is of class $\mathscr{C}_J(\theta;\bar{A})$ if and only if we have

$$\|a\|_X\leqslant C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}. \tag{1}$$

In fact, if X is of class $\mathscr{C}_J(\theta;\bar{A})$ we have that

$$\|a\|_X\leqslant C\max(t^{-\theta}\|a\|_{A_0}, t^{1-\theta}\|a\|_{A_1}),$$

for all $t>0$. Taking $t=\|a\|_{A_0}/\|a\|_{A_1}$, we get (1). Conversely, if (1) holds we see that

$$\|a\|_X\leqslant Ct^{-\theta}\|a\|_{A_0}^{1-\theta}(t\|a\|_{A_1})^{\theta}\leqslant Ct^{-\theta}J(t, a;\bar{A}).$$

Another useful formulation of the definition is given in the following theorem.

3.5.2. Theorem. *Suppose that* $0<\theta<1$. *Then*

(a) *X is of class $\mathscr{C}_K(\theta,\bar{A})$ iff*

$$\Delta(\bar{A})\subset X\subset\bar{A}_{\theta,\infty}.$$

(b) *A Banach space X is of class $\mathscr{C}_J(\theta,\bar{A})$ iff*

$$\bar{A}_{\theta,1}\subset X\subset\Sigma(\bar{A}).$$

In this theorem, we are, of course, only dealing with intermediate spaces (cf. Definition 3.5.1).

Proof: By the definition of $\bar{A}_{\theta,\infty}$ we have $X\subset\bar{A}_{\theta,\infty}$ if and only if

$$\sup_{t>0}t^{-\theta}K(t, a;\bar{A})\leqslant C\|a\|_X.$$

This clearly proves (a). In order to prove (b), we assume that $a=\sum_\nu u_\nu$ in $\Sigma(\bar{A})$. Then if X is a Banach space of class $\mathscr{C}_J(\theta;\bar{A})$

$$\|a\|_X\leqslant\sum_{-\infty}^{\infty}\|u_\nu\|_X\leqslant C\sum_{-\infty}^{\infty}2^{-\nu\theta}J(2^\nu, u_\nu;\bar{A}),$$

i.e.

$$\bar{A}_{\theta,1} \subset X.$$

Conversely, if this inclusion holds we put

$$u_\nu = \begin{cases} a & \text{if} \quad \nu = n, \\ 0 & \text{if} \quad \nu \neq n. \end{cases}$$

Then

$$\|a\|_X \leqslant C \, \|a\|_{\bar{A}_{\theta,1}} \leqslant C \, 2^{-n\theta} J(2^n, a; \bar{A}),$$

which shows that X is of class $\mathscr{C}_J(\theta; \bar{A})$. □

We are now ready to prove the reiteration theorem, which is one of the most important general results in interpolation theory. Often the reiteration theorem is called the stability theorem.

3.5.3. Theorem (The reiteration theorem). *Let $\bar{A} = (A_0, A_1)$ and $\bar{X} = (X_0, X_1)$ be two compatible couples of normed linear spaces, and assume that X_i $(i = 0, 1)$ are complete and of class $\mathscr{C}(\theta_i; \bar{A})$, where $0 \leqslant \theta_i \leqslant 1$ and $\theta_0 \neq \theta_1$.*

Put

$$\theta = (1 - \eta)\theta_0 + \eta\theta_1 \qquad (0 < \eta < 1).$$

Then, for $1 \leqslant q \leqslant \infty$

$$\bar{X}_{\eta,q} = \bar{A}_{\theta,q} \qquad (\textit{equivalent norms}).$$

In particular, if $0 < \theta_i < 1$ and $\bar{A}_{\theta_i,q_i}$ are complete then

$$(\bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_1,q_1})_{\eta,q} = \bar{A}_{\theta,q} \qquad (\textit{equivalent norms}).$$

Proof: Suppose that $a = a_0 + a_1 \in \bar{X}_{\eta,q}$ with $a_i \in X_i$. Since X_i is of class $\mathscr{C}(\theta_i; \bar{A})$, we have

$$K(t, a; \bar{A}) \leqslant K(t, a_0; \bar{A}) + K(t, a_1; \bar{A}) \leqslant C(t^{\theta_0} \|a_0\|_{X_0} + t^{\theta_1} \|a_1\|_{X_1}).$$

It follows that

$$K(t, a; \bar{A}) \leqslant C t^{\theta_0} K(t^{\theta_1 - \theta_0}, a; \bar{X}).$$

Applying $\Phi_{\theta,q}$ we deduce that

$$\Phi_{\theta,q}(K(t, a; \bar{A})) \leqslant C(\textstyle\int_0^\infty (t^{-(\theta-\theta_0)} K(t^{\theta_1-\theta_0}, a; X))^q \, dt/t)^{1/q}.$$

If we change the variable in the integral, writing $s=t^{\theta_1-\theta_0}$ and noting that $\eta=(\theta-\theta_0)/(\theta_1-\theta_0)$, we obtain

$$\Phi_{\theta,q}(K(t,a;\bar{A}))\leqslant C\Phi_{\eta,q}(K(s,a;\bar{X})).$$

(On the right hand side $\Phi_{\eta,q}$ is acting on the variable s.) It follows that $\bar{X}_{\eta,q}\subset\bar{A}_{\theta,q}$.

Next, we prove the reverse inclusion. Assume that $a\in\bar{A}_{\theta,q}$ and choose a representation $a=\int_0^\infty u(t)\,dt/t$ of a in $\Sigma(\bar{A})$. If $a\in\bar{X}_{\eta,q}$ we have, as above,

$$\|a\|^q_{\bar{X}_{\eta,q}}=C\int_0^\infty(t^{-(\theta-\theta_0)}K(t^{\theta_1-\theta_0},a;\bar{X})^q\,dt/t.$$

Using Lemma 3.2.1 and that X_i is of class $\mathscr{C}(\theta_i,\bar{A})$ we obtain

$$\begin{aligned}t^{\theta_0}K(t^{\theta_1-\theta_0},a;\bar{X})&\leqslant\int_0^\infty t^{\theta_0}K(t^{\theta_1-\theta_0},u(s);\bar{X})\,ds/s\\&\leqslant\int_0^\infty t^{\theta_0}\min(1,(t/s)^{\theta_1-\theta_0})J(s^{\theta_1-\theta_0},u(s);\bar{X})\,ds/s\\&\leqslant C\int_0^\infty\min((t/s)^{\theta_0},(t/s)^{\theta_1})J(s,u(s);\bar{A})\,ds/s.\end{aligned}$$

Changing the variable by putting $s=\sigma t$ and applying $\Phi_{\theta q}$, it follows that

$$\|a\|_{\bar{X}_{\eta,q}}\leqslant C(\int_0^\infty\sigma^\theta\min(\sigma^{-\theta_0},\sigma^{-\theta_1})\,d\sigma/\sigma)\,\Phi_{\theta,q}(J(s,u(s);\bar{A}),$$

by Lemma 3.2.1. Since the integral is finite, the inclusion is established by taking the infimum in view of the equivalence Theorem 3.3.1. □

In the case $\theta_0=\theta_1$ we have the following complement to the reiteration theorem.

3.5.4. Theorem. *Let $\bar{A}$ be a given couple of Banach spaces and put*

$$X_0=\bar{A}_{\theta,q_0},\qquad X_1=\bar{A}_{\theta,q_1}$$

where $0<\theta<1, 1\leqslant q_i\leqslant\infty$ $(i=0,1)$. *Then*

$$\bar{X}_{\eta,q}=\bar{A}_{\theta,q}$$

where

$$\frac{1}{q}=\frac{1-\eta}{q_0}+\frac{\eta}{q_1}.$$

The proof of this theorem will be given in 5.2 (Theorem 5.2.4).

3.6. A Formula for the K-Functional

By the reiteration theorem, we have $\bar{X}_{\eta,q}=\bar{A}_{\theta,q}$ if $\bar{X}=(\bar{A}_{\theta_0,q_0},\bar{A}_{\theta_1,q_1})$. This suggests the possibility of a formula connecting the functional $K(t,a;\bar{A})$ with $K(t,a;\bar{X})$. Such a formula was given by Holmstedt [1].

3.6.1. Theorem. *Let $\bar{A}$ be a given couple of normed spaces and put*

$$X_0=\bar{A}_{\theta_0,q_0},\qquad X_1=\bar{A}_{\theta_1,q_1},$$

where $0\leqslant\theta_0<\theta_1\leqslant 1$ and $1\leqslant q_0\leqslant\infty$, $1\leqslant q_1\leqslant\infty$. Put $\lambda=\theta_1-\theta_0$. Then

$$\begin{aligned}K(t,a;\bar{X})\sim&\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{q_0}\,ds/s\right)^{1/q_0}\\&+t\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_1}K(s,a;\bar{A}))^{q_1}\,ds/s\right)^{1/q_1}.\end{aligned}$$

Proof: We first prove "$\geqslant$". Let $a=a_0+a_1$, $a_i\in A_i$, $i=0,1$. By Theorem 3.1.2 and Minkowski's inequality it follows that

$$\begin{aligned}\left(\int_0^{t^{1/\lambda}}(s^{-\theta}K(s,a;\bar{A}))^{q_0}\,ds/s\right)^{1/q_0}&\leqslant\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a_0;\bar{A}))^{q_0}\,ds/s\right)^{1/q_0}\\&\quad+\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a_1;\bar{A}))^{q_0}\,ds/s\right)^{1/q_0}\\&\leqslant\|a\|_{X_0}+C\left(\int_0^{t^{1/\lambda}}(s^{\lambda}\|a_1\|_{X_1})^{q_0}\,ds/s\right)^{1/q_0}\leqslant C(\|a_0\|_{X_0}+t\|a_1\|_{X_1}).\end{aligned}$$

Similarly, we obtain

$$t\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_1}K(s,a;\bar{A}))^{q_1}\,ds/s\right)^{1/q_1}\leqslant C(\|a_0\|_{X_0}+t\|a_1\|_{X_1}).$$

Adding the estimates and taking the infimum, the proof of "$\geqslant$" is complete.

We turn to the proof of "$\leqslant$". By the definition of $K(t,a;\bar{A})$, we may choose $a_0(t)\in A_0$ and $a_1(t)\in A_1$ such that $a=a_0(t)+a_1(t)$ and

$$\|a_0(t)\|_{A_0}+t\|a_1(t)\|_{A_1}\leqslant 2K(t,a;\bar{A}).$$

With this choice we have

$$\begin{aligned}K(t,a;\bar{X})&\leqslant\|a_0(t^{1/\lambda})\|_{X_0}+t\|a_1(t^{1/\lambda})\|_{X_1}\\&=\left(\int_0^{\infty}(s^{-\theta_0}K(s,a_0(t^{1/\lambda});\bar{A}))^{q_0}\,ds/s\right)^{1/q_0}\\&\quad+t\left(\int_0^{\infty}(s^{-\theta_1}K(s,a_1(t^{1/\lambda});\bar{A}))^{q_1}\,ds/s\right)^{1/q_1}\\&\leqslant\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a_0(t^{1/\lambda});\bar{A}))^{q_0}\,ds/s\right)^{1/q_0}\\&\quad+\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_0}K(s,a_0(t^{1/\lambda});\bar{A}))^{q_0}\,ds/s\right)^{1/q_0}\\&\quad+t\left(\int_0^{t^{1/\lambda}}(s^{-\theta_1}K(s,a_1(t^{1/\lambda});\bar{A}))^{q_1}\,ds/s\right)^{1/q_1}\\&\quad+t\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_1}K(s,a_1(t^{1/\lambda});\bar{A}))^{q_1}\,ds/s\right)^{1/q_1}.\end{aligned}$$

We estimate each term separately, using Lemma 3.1.1. For the first term we obtain, by the triangle inequality,

$$(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a_0(t^{1/\lambda});\bar{A}))^{q_0}ds/s)^{1/q_0}\leqslant(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{q_0}ds/s)^{1/q_0}$$
$$+(\int_0^{t^{1/\lambda}}(s^{-\theta}K(s,a_1(t^{1/\lambda});\bar{A}))^{q_0}ds/s)^{1/q_0},$$

where the last term is bounded by

$$(\int_0^{t^{1/\lambda}}(s^{-\theta_0}s\,\|a_1(t^{1/\lambda})\|_{A_1})^{q_0}ds/s)^{1/q_0}\leqslant Ct^{-1/\lambda}K(t^{1/\lambda},a;\bar{A})t^{(1-\theta_0)/\lambda}$$
$$\leqslant C(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{q_0}ds/s)^{1/q_0};$$

the last inequality holds since $s^{-1}K(s,a)$ is decreasing.

To estimate the second term, we similarly infer that

$$(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_0}K(s,a_0(t^{1/\lambda});\bar{A}))^{q_0}ds/s)^{1/q_0}\leqslant(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_0}\,\|a_0(t^{1/\lambda})\|_{A_0})^{q_0}ds/s)^{1/q_0}$$
$$\leqslant Ct^{-\theta_0/\lambda}\,\|a_0(t^{1/\lambda})\|_{A_1}\leqslant Ct^{-\theta_0/\lambda}K(t^{1/\lambda},a;\bar{A})$$
$$\leqslant C(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{q_0}ds/s)^{1/q_0}.$$

The third and fourth terms are treated analogously. Summing the four estimates, we get "$\leqslant$". □

The following corollary is easily proved by an adaptation of the above proof; we leave this as an exercise.

3.6.2. Corollary. *Let $\bar{A}$ be a given couple of normed spaces.*

a) *Put* $\bar{X}=(A_0,A_{\theta_1,q_1})$, $\lambda=\theta_1$. *Then*

$$K(t,a;\bar{X})\sim t(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_1}K(s,a;\bar{A}))^{q_1}ds/s)^{1/q_1}.$$

b) *Put* $\bar{X}=(\bar{A}_{\theta_0,q_0},A_1)$, $\lambda=1-\theta_0$. *Then*

$$K(t,a;\bar{X})\sim(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{q_0}ds/s)^{1/q_0}.\quad\square$$

3.7. The Duality Theorem

We consider the category $\mathscr{B}$ of all Banach spaces. Here we determine the dual $\bar{A}'_{\theta,q}$ of the interpolation space $\bar{A}_{\theta,q}$ when $1\leqslant q<\infty$. Recall that if $\Delta(\bar{A})$ is dense in A_0 and in A_1 we have

$$K(t,a';A'_0,A'_1)=\sup_{a\in\Delta(\bar{A})}\frac{|\langle a',a\rangle|}{J(t^{-1},a;A_0,A_1)}\tag{1}$$

and

(2) $$J(t, a'; A_0', A_1') = \sup_{a \in \Delta(\bar{A})} \frac{|\langle a', a\rangle}{K(t^{-1}, a; A_0, A_1)}.$$

(1) and (2) are immediate consequences of Theorem 2.7.1.

These formulas suggest a simple relation between the space $\bar{A}_{\theta,q}$ and its dual. By Theorem 3.4.2, we know that $\Delta(\bar{A})$ is dense in $\bar{A}_{\theta,q}$ if $q<\infty$. Since

$$\Delta(\bar{A}) \subset \bar{A}_{\theta,q} \subset \Sigma(\bar{A})$$

(dense inclusions) we have, for $q<\infty$,

$$\Delta(\bar{A}') \subset (\bar{A}_{\theta,q})' \subset \Sigma(\bar{A}').$$

We shall now prove the following result.

3.7.1. Theorem (The duality theorem). *Let $\bar{A}$ be a couple of Banach spaces, such that $\Delta(\bar{A})$ is dense in A_0 and A_1. Assume that $1 \leqslant q < \infty$ and $0<\theta<1$. Then*

$$(A_0, A_1)'_{\theta,q} = (A_0', A_1')_{\theta,q'} \quad \textit{(equivalent norms)},$$

where $1/q + 1/q' = 1$.

Proof: We shall prove that

(3) $$(A_0, A_1)'_{\theta,q;J} \subset (A_1', A_0')_{1-\theta,q';K}$$

(4) $$(A_0, A_1)'_{\theta,q;K} \supset (A_1', A_0')_{1-\theta,q';J}.$$

Using (3) and (4), we get the result by the Equivalence Theorem 3.3.1 and Theorem 3.4.1.

In order to prove (3), we take $a' \in (A_0, A_1)'_{\theta,q;J}$, and apply Formula (1). Thus, given $\varepsilon>0$, we can find $b_\nu \in \Delta(\bar{A})$ such that $b_\nu \neq 0$ and, since $a' \in \Delta(\bar{A})' = \Sigma(\bar{A}')$,

$$K(2^{-\nu}, a'; A_0', A_1') - \varepsilon \min(1, 2^{-\nu}) \leqslant (J(2^\nu, b_\nu; A_0, A_1))^{-1} \langle a', b_\nu\rangle.$$

Choose a sequence $\alpha \in \lambda^{\theta,q}$, and put

$$a_\alpha = \sum_\nu (J(2^\nu, b_\nu; A_0, A_1))^{-1} \alpha_\nu \cdot b_\nu.$$

It follows that $a_\alpha \in (A_0, A_1)_{\theta,q;J}$,

$$\langle a', a_\alpha\rangle \geqslant \sum_\nu (K(2^{-\nu}, a'; A_0', A_1') - \varepsilon \min(1, 2^{-\nu}))$$

and

$$\langle a', a_\alpha\rangle \leqslant \|\alpha\|_{\lambda^{\theta,q}} \|a'\|_{(A_0,A_1)'_{\theta,q;J}},$$

since $\|\alpha\|_{\lambda^{\theta,q}} \geqslant \|a_\alpha\|_{\theta,q;J}$. Noting that $K(2^{-\nu}, a'; A'_0, A'_1) = 2^{-\nu} K(2^\nu, a'; A'_1, A'_0)$, we obtain

$$\sum_\nu 2^{-\nu}\alpha_\nu (K(2^\nu, a'; A'_1, A'_0) - \varepsilon \min(1, 2^\nu))$$
$$\leqslant \|\alpha\|_{\lambda^{\theta,q}} \cdot \|a'\|_{(A_0,A_1)'_{\theta,q;J}}.$$

Since $\lambda^{\theta,q}$ and $\lambda^{1-\theta,q'}$ are dual via the duality $\sum_\nu 2^{-\nu}\alpha_\nu \beta_\nu$ and ε is arbitrary, (3) follows.

In order to prove (4), we take an element a' in $(A'_1, A'_0)_{1-\theta,q';J}$. We write a' as a sum

$$a' = \sum_\nu a'_\nu$$

with convergence in $\Sigma(\bar{A}') = \Delta(\bar{A})'$. Then it follows that

$$|\langle a', a\rangle| \leqslant \sum_\nu |\langle a'_\nu, a\rangle| \leqslant \sum_\nu J(2^{-\nu}, a'_\nu; A'_0, A'_1) K(2^\nu, a; A_0, A_1).$$

Since

$$J(2^{-\nu}, a'_\nu; A'_0, A'_1) = 2^{-\nu} J(2^\nu, a'_\nu, A'_1, A'_0)$$

we conclude that

$$|\langle a', a\rangle| \leqslant \sum_\nu 2^{-\nu} J(2^\nu, a'_\nu; A'_1, A'_0) K(2^\nu, a; A_0, A_1)$$

which implies (4). □

Remark: In the case $q = \infty$ we see from the proof above that if $\bar{A}^0_{\theta,\infty}$ denotes the closure of $\Delta(\bar{A})$ in $\bar{A}_{\theta,\infty}$ then

$$(\bar{A}^0_{\theta,\infty})' = \bar{A}'_{\theta,1}.$$

3.8. A Compactness Theorem

Using Theorem 3.4.1, we see that if $A_1 \subset A_0$ then $\bar{A}_{\theta_1,q_1} \subset A_{\theta_0,q_1}$ when $\theta_0 < \theta_1$, and $\bar{A}_{\theta,q} \subset \bar{A}_{\theta,r}$ when $q \leqslant r$. It follows that

(1) $$\bar{A}_{\theta_1,q_1} \subset \bar{A}_{\theta_0,q_0} \quad \text{if} \quad \theta_0 < \theta_1.$$

If the inclusion $A_1 \subset A_0$ is compact, then so is the inclusion (1). This will follow from our next theorem.

3.8.1. Theorem. *Let B be any Banach space and (A_0, A_1) a couple of Banach spaces. Let T be a linear operator.*

(i) *Assume that*

$$T: A_0 \to B \quad \textit{compactly},$$
$$T: A_1 \to B,$$

and that E is of class $\mathscr{C}_K(\theta; \bar{A})$ for some θ with $0<\theta<1$. Then

$$T: E \to B \quad \textit{compactly}.$$

(ii) *Assume that*

$$T: B \to A_0 \quad \textit{compactly},$$
$$T: B \to A_1,$$

and that E is of class $\mathscr{C}_J(\theta; \bar{A})$ for some θ with $0<\theta<1$. Then

$$T: B \to E \quad \textit{compactly}.$$

Proof: (i) Let $(a_\nu)_1^\infty$ be a bounded sequence in E and assume that $\|a_\nu\|_E \leqslant 1$. Moreover let M_j be the norm of T as a mapping from A_j to B. For a given $\varepsilon>0$ we choose t so that $t^\theta<\varepsilon t$. Next we choose $a_{\nu 0}\in A_0$ and $a_{\nu 1}\in A_1$, such that $a_\nu = a_{\nu 0}+a_{\nu 1}$ and

$$\|a_{\nu 0}\|_{A_0} + t\,\|a_{\nu 1}\|_{A_1} \leqslant 2\,K(t, a_\nu; \bar{A}).$$

By the assumption on E we have $K(t, a_\nu; \bar{A}) \leqslant Ct^\theta \|a_\nu\|_E$. It follows that

$$\|a_{\nu 0}\|_{A_0} + t\,\|a_{\nu 1}\|_{A_1} \leqslant 2Ct^\theta \|a_\nu\|_E \leqslant 2Ct^\theta.$$

Thus $(a_{\nu 0})_1^\infty$ is bounded in A_0. Since T is a compact operator from A_0 into B we can find a subsequence $(a_{\nu' 0})$ of $(a_{\nu 0})_1^\infty$ so that

$$\|Ta_{\nu' 0} - Ta_{\mu' 0}\|_B \leqslant \varepsilon,$$

if ν', μ' are large enough. Since

$$\|Ta_{\nu' 1} - Ta_{\mu' 1}\|_B \leqslant M_1\,\|a_{\nu' 1} - a_{\mu' 1}\|_{A_1} \leqslant 2CM_1 t^{\theta-1} \leqslant 2CM_1\varepsilon,$$

we conclude that

$$\|Ta_{\nu'} - Ta_{\mu'}\|_B \leqslant \varepsilon(1+2CM_1)$$

if ν', μ' are large enough. This proves the compactness of the operator $T: E \to B$.

(ii) Let $(b_\nu)_1^\infty$ be a bounded sequence in B with $\|b_\nu\|_B \leqslant 1$, and let M_j be the norm of T as a mapping from B to A_j. Given an $\varepsilon > 0$ we choose t so that $t < \varepsilon t^\theta$. Passing to a subsequence we may assume that

$$\|Tb_{\nu'} - Tb_{\mu'}\|_{A_0} \leqslant t$$

if ν', μ' are large enough. Moreover we have

$$\|Tb_{\nu'} - Tb_{\mu'}\|_{A_1} \leqslant 2M_1.$$

By the assumption on E we have that $t^\theta \|a\|_E \leqslant C J(t, a; \bar{A})$. Thus we conclude that

$$t^\theta \|Tb_{\nu'} - Tb_{\mu'}\|_E \leqslant C J(t, Tb_{\nu'} - Tb_{\mu'}; \bar{A}) \leqslant C \max(1, 2M_1) t.$$

Hence we see that, with a new constant C,

$$t^\theta \|Tb_{\nu'} - Tb_{\mu'}\|_E \leqslant C\varepsilon t^\theta,$$

i.e.

$$\|Tb_{\nu'} - Tb_{\mu'}\|_E \leqslant C\varepsilon.$$

Therefore $T: B \to E$ is compact. $\square$

3.8.2. Corollary. *If A_0 and A_1 are Banach spaces, $A_1 \subset A_0$ with compact inclusion and $0 < \theta_0 < \theta_1 < 1$ then $\bar{A}_{\theta_1, q_1} \subset \bar{A}_{\theta_0, q_0}$ with compact inclusion.*

Proof: We use part (i) of Theorem 3.8.1 on the identity mapping I. By assumption, $I: A_1 \to A_0$ compactly. It is trivial that $I: A_0 \to A_0$ and thus I maps the space $\bar{A}_{\theta_1, q_1}$ compactly to A_0. Thus

$$\bar{A}_{\theta_1, q_1} \subset A_0 \quad \textit{(compact inclusion)}.$$

Using part (ii), we get in the same way

$$\bar{A}_{\theta_1, q_1} \subset (A_0, \bar{A}_{\theta_1, q_1})_{\eta, q_0} \quad \textit{(compact inclusion)}.$$

By Theorem 3.5.3, we see that

$$(A_0, A_{\theta_1, q_1})_{\eta, q_0} = \bar{A}_{\theta_0, q_0} \quad \text{if} \quad \theta_0 = \eta\theta_1.$$

Now the result follows. $\square$

3.9. An Extremal Property of the Real Method

In this section we shall prove that the interpolation functors $J_{\theta,1}$ and $K_{\theta,\infty}$ are extremal in the sense explained in the following theorem.

3.9.1. Theorem. *Suppose that F is an interpolation functor of exponent θ. Then, for any compatible Banach couple $\bar{A}=(A_0, A_1)$, we have*

$$J_{\theta,1}(\bar{A}) \subset F(\bar{A}).$$

Moreover, if $\Delta(\bar{A})$ is dense in A_0 and in A_1 then

$$F(\bar{A}) \subset K_{\theta,\infty}(\bar{A}).$$

Proof: Write $A=F(\bar{A})$ and consider the mapping

$$T\lambda=\lambda a,$$

where a is a given element in A and $\lambda \in \mathbb{C}$. Clearly $T: \mathbb{C} \to A_j$ with norm $\|a\|_{A_j}$. Thus $T: \mathbb{C} \to F(\bar{A})$ with norm less than a constant multiplied by $\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}$. It follows that

$$\|a\|_A \leqslant C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta},$$

or, equivalently,

$$\|a\|_A \leqslant C 2^{-\nu\theta} J(2^\nu, a; \bar{A}). \tag{1}$$

If $a=\sum u_\nu$ in $\Sigma(\bar{A})$ we therefore obtain

$$\|a\|_A \leqslant C \textstyle\sum_\nu 2^{-\nu\theta} J(2^\nu, u_\nu; \bar{A}),$$

i.e.

$$J_{\theta,1}(\bar{A}) \subset A.$$

In order to prove the inclusion $A \subset K_{\theta,\infty}(\bar{A})$, we take $a' \in \Delta(\bar{A}')$ and put $Ta=\langle a', a\rangle$. Then $T: A_i \to \mathbb{C}$ with norm $\|a'\|_{A'_i}$ $(i=0,1)$. By the assumptions it follows that $T: A \to \mathbb{C}$ with norm

$$\sup_{a\in A} |\langle a', a\rangle|/\|a\|_A \leqslant C\|a'\|_{A'_0}^{1-\theta}\|a'\|_{A'_1}^{\theta} \leqslant C t^\theta J(t^{-1}, a'; \bar{A}').$$

Thus for all $a \in A$ and all $a' \in \Delta(\bar{A}')$ we have

$$|\langle a', a\rangle|/J(t^{-1}, a'; \bar{A}') \leqslant C t^\theta \|a\|_A.$$

Noting that $\Sigma(\bar{A})' = \Delta(\bar{A}')$ (Theorem 2.7.1) and taking the supremum over all $a' \in \Delta(\bar{A}')$ on the left hand side, we conclude that

$$K(t, a; \bar{A}) \leqslant C t^\theta \|a\|_A,$$

which means that $A \subset K_{\theta,\infty}(\bar{A})$. $\square$

3.10. Quasi-Normed Abelian Groups

The development of the real interpolation method did not depend heavily on all the properties of a norm. It is easily seen that the homogeneity $\|\lambda a\| = |\lambda|\,\|a\|$ is not used. In several contexts the triangle inequality can be replaced by the more general quasitriangle inequality $\|a+b\| \leqslant k(\|a\| + \|b\|)$. This indicates the possibility of extending the real interpolation method to more general categories of spaces. Such an extension is motivated by certain applications, for instance in order to get the full version of the Marcinkiewicz interpolation theorem. In this and the next two sections, we shall extend the real interpolation method to quasi-normed Abelian groups.

Let A be an Abelian group. The group operation is denoted by $+$, the inverse of a is $-a$ and the neutral element is 0. A *quasi-norm* on A is a real-valued function $\|\cdot\|_A$, defined on A, such that

(1) $\qquad \|a\|_A \geqslant 0, \quad \text{and} \quad \|a\|_A = 0 \quad \text{iff} \quad a=0,$

(2) $\qquad \|-a\|_A = \|a\|_A,$

(3) $\qquad \|a+b\|_A \leqslant c(\|a\|_A + \|b\|_A),$

where $c \geqslant 1$. The inequality (3) is called the *c-triangle inequality* and the function $\|\cdot\|_A$ a *c-norm.*

The topology on A is defined in a natural way. A basis for the neighbourhoods is the collection of all sets $\{b : \|b-a\|_A < \varepsilon\}$ where $\varepsilon > 0$. When $c=1$ the topology is defined by means of the metric $d(a,b) = \|b-a\|_A$. From the following lemma, we see that A is metrizable also in the case $c > 1$.

3.10.1. Lemma. *Suppose that A is a c-normed Abelian group and let ρ be defined by the equation $(2c)^\rho = 2$. Then there is a 1-norm $\|\cdot\|_A^*$ on A, such that*

(4) $\qquad \|a\|_A^* \leqslant \|a\|_A^\rho \leqslant 2\,\|a\|_A^*.$

It follows that $d(a,b) = \|b-a\|_A^*$ is a metric defining the topology in A.

Proof: We define $\|a\|_A^*$ by the formula

$$\|a\|_A^* = \inf\{\textstyle\sum_{j=1}^n \|a_j\|_A^\rho : \ \sum_{j=1}^n a_j = a,\ n \geqslant 1\}.$$

Taking $n=1$ and $a_1 = a$ we see that $\|a\|_A^* \leqslant \|a\|_A^\rho$. It is also easy to see that $\|a\|_A^*$ is a 1-norm. In fact, if $a = a_1 + \dots + a_n$ and $b = b_1 + \dots + b_m$ and $c = a+b$, we put $c_j = a_j$ if $1 \leqslant j \leqslant n$ and $c_j = b_{j-n}$ if $n+1 \leqslant j \leqslant n+m$. Then $c = c_1 + \dots + c_{n+m}$ and

$$\|c\|_A^* \leqslant \textstyle\sum_{j=1}^{n+m} \|c_j\|_A^\rho = \sum_{j=1}^n \|a_j\|_A^\rho + \sum_{j=1}^m \|b_j\|_A^\rho.$$

This implies the 1-triangle inequality

$$\|c\|_A^* \leqslant \|a\|_A^* + \|b\|_A^*.$$

Obviously (2) is also satisfied.

It remains to prove $\|a\|_A^\rho \leqslant 2\,\|a\|_A^*$. (Note that this inequality also implies (1).) First we note that

$$\|\textstyle\sum_{j=1}^n a_j\|_A^\rho \leqslant \max_{1 \leqslant j \leqslant n}(2^{v_j}\|a_j\|_A^\rho), \qquad n \geqslant 1, \tag{5}$$

when $v_1, \ldots, v_n$ are any integers such that

$$v_j \geqslant 0, \quad \textstyle\sum_{j=1}^n 2^{-v_j} \leqslant 1. \tag{6}$$

In fact, (5) is true if $n=1$. Assume that (5) holds for $1, 2, \ldots, n-1$. Considering (6), it is easily seen that there are two disjoint, nonempty sets I_1 and I_2, such that $I_1 \cup I_2 = \{1, \ldots, n\}$ and

$$\textstyle\sum_{j \in I_1} 2^{-v_j+1} \leqslant 1, \qquad \sum_{j \in I_2} 2^{-v_j+1} \leqslant 1.$$

By the induction hypothesis, it follows that

$$\|\textstyle\sum_{j \in I_k} a_j\|_A^\rho \leqslant \max_{j \in I_k} 2^{v_j - 1}\|a_j\|_A^\rho, \qquad k = 1, 2.$$

Consequently,

$$\|\textstyle\sum_{j=1}^n a_j\|_A^\rho \leqslant \max(2\,\|\sum_{j \in I_1} a_j\|_A^\rho, 2\,\|\sum_{j \in I_2} a_j\|_A^\rho) \leqslant \max_{j=1,\ldots,n}(2^{v_j}\|a_j\|_A^\rho).$$

Thus we have proved (5).

Now suppose that $a = a_1 + \cdots + a_n$ and put

$$M = \textstyle\sum_{j=1}^n \|a_j\|_A^\rho.$$

Choose $v_1, \ldots, v_n$ so that

$$2^{-v_j} \leqslant \|a_j\|_A^\rho / M \leqslant 2^{-v_j+1}.$$

Then (6) holds, and thus by (5) we have

$$\|a\|_A^\rho \leqslant \max_{1 \leqslant j \leqslant n}(2^{v_j}\|a_j\|_A^\rho) \leqslant 2M.$$

Since $\|a\|_A$ is the infimum of all M, we obtain $\|a\|_A^\rho \leqslant 2\,\|a\|_A^*$. □

Since every quasi-normed Abelian group is metrizable, we have the notions of Cauchy sequences and completeness. It is easy to verify the following analogue of Lemma 2.2.1. (We leave the proof to the reader.)

3.10.2. Lemma. *Suppose that A is a c-normed Abelian group and let ρ be defined by the equation $(2c)^\rho = 2$. If $a = \sum_{j=0}^{\infty} a_j$ converges in A, then*

$$\|a\|_A \leqslant C(\sum_{j=0}^{\infty} \|a_j\|_A^\rho)^{1/\rho}.$$

Moreover, if A is complete then the finiteness of the right hand side implies the convergence in A of the series $\sum_{j=0}^{\infty} a_j$. □

We shall now give an example which will be used in the next section and in Chapter 7.

Example: Suppose that $0 < p < \infty$ and let μ be any positive measure on a measure space (U, μ). Let $L_p = L_p(d\mu)$ denote the space of μ-measurable functions f, such that

$$\int_U |f|^p \, d\mu < \infty.$$

In the limiting case $p = \infty$ we get the space $L_\infty = L_\infty(d\mu)$ of all bounded μ-measurable functions. Let us write

$$\|f\|_{L_p} = (\int_U |f|^p \, d\mu)^{1/p} \quad \text{if} \quad 0 < p < \infty$$

and

$$\|f\|_{L_\infty} = \operatorname{ess\,sup} |f(x)|.$$

Note that L_p is a vector space, but for the moment we forget about multiplication by scalars. Thus we consider L_p as an Abelian group.

3.10.3. Lemma. *The Abelian group L_p is c-normed with $c = 1$ if $1 \leqslant p \leqslant \infty$ and $c = 2^{(1-p)/p}$ if $0 < p < 1$. Thus we have*

$$\|f+g\|_{L_p} \leqslant \max(1, 2^{(1-p)/p})(\|f\|_{L_p} + \|g\|_{L_p}). \tag{7}$$

Moreover, L_p is complete.

Proof: We consider only the case $0 < p < 1$ since the case $1 \leqslant p \leqslant \infty$ is covered by the familiar theory of L_p-spaces. In order to prove (7), we shall use the well-known inequalities $(x \geqslant 0,\ y \geqslant 0)$

$$(x+y)^p \leqslant x^p + y^p \leqslant 2^{1-p}(x+y)^p \quad \text{if} \quad 0 < p < 1.$$

From the left hand inequality we obtain

$$\|f+g\|_{L_p} \leqslant (\int_U (|f| + |g|)^p \, d\mu)^{1/p} \leqslant (\|f\|_{L_p}^p + \|g\|_{L_p}^p)^{1/p}.$$

From the right hand inequality we now obtain

$$\|f+g\|_{L_p} \leqslant 2^{(1-p)/p}(\|f\|_{L_p} + \|g\|_{L_p}).$$

In order to prove completeness, we shall use Lemma 2.2.1. If $(2c)^\rho = 2$ and $c = 2^{(1-p)/p}$ we have $\rho = p$. Therefore we assume that

$$\sum_{j=1}^{\infty} \|f_j\|_{L_p}^p < \infty .$$

Thus $\sum_{j=1}^{\infty} |f_j|^p$ converges in L_1. Put $f = \sum_{j=1}^{\infty} f_j$. Then f is measurable and

$$|f|^p \leqslant \sum_{j=1}^{\infty} |f_j|^p.$$

Thus $|f|^p$ belongs to L_1, i.e. $f \in L_p$. It follows that $f = \sum_{j=1}^{\infty} f_j$ converges in L_p. □

Consider the case $0 < p \leqslant 1$. In the notation of Lemma 3.10.1, we then have $c = 2^{(1-p)/p}$ and $\rho = p$. Thus

$$\|f\|_{L_p}^* = \int_U |f|^p \, d\mu.$$

defines an equivalent norm on L_p. Following Peetre-Sparr [1], we can now consider the limiting case $p = 0$. Let $\operatorname{supp} f$ denote the support of f, i.e. any measurable set E, such that $f = 0$ outside E and $f \neq 0$ almost everywhere on E. Clearly E is unique up to sets of measure 0. Then

$$\lim_{p \to 0} \|f\|_{L_p}^* = \int_E d\mu = \mu(\operatorname{supp} f).$$

We thus define L_0 to be the space of all measurable functions f, such that

$$\|f\|_{L_0} = \mu(\operatorname{supp} f) < \infty.$$

Since $\operatorname{supp}(f+g) \subset (\operatorname{supp} f) \cup (\operatorname{supp} g)$, we see that L_0 is a 1-normed space.

Note that if $0 < p \leqslant 1$ we have

$$\|\lambda f\|_{L_p} = |\lambda| \, \|f\|_{L_p},$$

$$\|\lambda f\|_{L_p}^* = |\lambda|^p \, \|f\|_{L_p}^*.$$

Moreover we have

$$\|\lambda f\|_{L_0} = \|f\|_{L_0}, \qquad \lambda \neq 0$$

(since f and λf have the same support). □

The example above is typical. The quasi-normed Abelian groups we shall consider in the sequel are in fact vector spaces, where the quasi-norm is not homogeneous. In a quasi-normed vector space we require not only the properties (1) and (3) but also

$$\|\lambda a\|_A = |\lambda| \, \|a\|_A, \qquad \lambda \text{ scalar}.$$

Now let A and B be two quasi-normed Abelian groups. Then a mapping T from A into B is called a *homomorphism* if $T(-a)=-T(a)$, $T(a+b)=T(a)+T(b)$. The homomorphism T is *bounded* if

$$\|T\|_{A,B}=\sup_{a\neq 0}\|Ta\|_B/\|a\|_A<\infty.$$

The bounded homomorphisms are continuous and constitute a quasi-normed Abelian group. We shall let $\mathscr{A}$ stand for the category of all quasi-normed Abelian groups, the morphisms being the bounded homomorphisms. Sometimes we shall also consider the category $\mathscr{Q}$ of all quasi-normed vector spaces. In $\mathscr{Q}$ the morphisms are the bounded linear mappings, i.e., bounded homomorphisms satisfying the additional assumption $T(\lambda a)=\lambda T(a)$. Moreover, the morphisms constitute a quasi-normed vector space.

The notion of *compatible spaces* A_0 and A_1 carries over without change. We also have the analogue of Lemma 2.3.1. In fact, if A_j is c_j-normed then it is easily seen that $\Delta(\bar{A})=A_0\cap A_1$ and $\Sigma(\bar{A})=A_0+A_1$ are c-normed spaces with $c=\max(c_0, c_1)$. If A_0 and A_1 are complete then so are $\Delta(\bar{A})$ and $\Sigma(\bar{A})$. (Use Lemma 3.10.2.)

If $\mathscr{C}$ is a sub-category of $\mathscr{A}$ we can form the category $\mathscr{C}_1$ of *compatible couples* $\bar{A}=(A_0, A_1)$. Here we can adopt the same conventions and notation as in Section 2.3. As a consequence the definitions of *intermediate space*, *interpolation space(s)* and *interpolation functor* carry over without change. (See Definition 2.4.1 and 2.4.5.)

3.11. The Real Interpolation Method for Quasi-Normed Abelian Groups

In this section we shall consider the category $\mathscr{A}$ of all quasi-normed Abelian groups. For any couple $\bar{A}$ in $\mathscr{A}_1$ we can define the functionals $K(t, a; \bar{A})$ and $J(t, a; \bar{A})$. Clearly Lemma 3.1.1 and 3.2.1 still hold. We can also imitate the definitions of the spaces $K_{\theta,q}(\bar{A})$ and $J_{\theta,q}(\bar{A})$ without changing anything. For $J_{\theta,q}(\bar{A})$ we use the discrete definition, which is equivalent to the continuous one in the category $\mathscr{N}$, in order to avoid integration in the quasi-normed groups.

In the case $0<\theta<1$ we can even extend the range of the parameter q, *allowing q to be any positive real number.* Then we still have Lemma 3.1.3 and Lemma 3.2.2. The interpolation theorems 3.1.2 and 3.2.3 are still true in the category $\mathscr{A}$. It should be noticed that $\|a\|_{\theta,q;K}$ and $\|a\|_{\theta,q;J}$ are no longer norms but merely quasi-norms. This will follow from the next lemma.

3.11.1. Lemma. *Suppose that A_j is c_j-normed. Then*

(1) $$K(t, a+b; \bar{A})\leqslant c_0(K(c_1t/c_0, a; \bar{A})+K(c_1t/c_0, b; \bar{A})),$$

and

(2) $$J(t, a+b; \bar{A})\leqslant c_0(J(c_1t/c_0, a; \bar{A})+J(c_1t/c_0, b; \bar{A})).$$

Proof: If $a=a_0+a_1$ and $b=b_0+b_1$ we have

$$K(t, a+b)\leqslant \|a_0+b_0\|_{A_0}+t\|a_1+b_1\|_{A_1}$$
$$\leqslant c_0(\|a_0\|_{A_0}+\|b_0\|_{A_0})+tc_1(\|a_1\|_{A_1}+\|b_1\|_{A_1})$$

which implies (1). The second estimate is equally easy to prove. □

Using Lemma 3.11.1 we are now able to prove the analogues of the interpolation theorems 3.1.2 and 3.2.3. Here we prefer to formulate the result in one theorem.

3.11.2. Theorem. *$K_{\theta,q}$ and $J_{\theta,q}$ are interpolation functors of exponent θ on the category of quasi-normed Abelian groups and $K_{\theta,q}$ is exact of exponent θ. Moreover, we have*

(3) $$K(s, a; \bar{A})\leqslant \gamma_{\theta,q}s^{\theta}\|a\|_{\theta,q;K},$$

where $0<\theta<1$, $0<q\leqslant\infty$, *or* $0\leqslant\theta\leqslant 1$, $q=\infty$; *and*

(4) $$\|a\|_{\theta,q;J}\leqslant Cs^{-\theta}J(s, a; \bar{A}).$$

Here C is independent of θ.

Proof: The proofs of Theorem 3.1.2 and 3.2.3 carry over without change as long as we do not use the triangle inequality. As far as the K-method is concerned, we shall therefore only have to prove that $A=K_{\theta,q}(\bar{A})$ is a quasi-normed Abelian group. This amounts to proving the quasi-triangle inequality.

Using Lemma 3.10.3, we see that

$$\Phi_{\theta,q}(\varphi+\psi)\leqslant \max(1, 2^{(1-q)/q})(\Phi_{\theta,q}(\varphi)+\Phi_{\theta,q}(\psi)).$$

Combining this inequality with Lemma 3.11.1, Formula (1), and with the equality

$$\Phi_{\theta,q}(\varphi(t/s))=s^{-\theta}\Phi_{\theta,q}(\varphi(t)),$$

we obtain

$$\|a+b\|_{\theta,q;K}\leqslant \max(1, 2^{(1-q)/q})c_0(c_1/c_0)^{\theta}(\|a\|_{\theta,q;K}+\|b\|_{\theta,q;K}).$$

Thus if A_j is c_j-normed, $j=0,1$, then $A=K_{\theta,q}(\bar{A})$ is c-normed with

$$c=c_0^{1-\theta}c_1^{\theta}\max(1, 2^{(1-q)/q}).$$

Using the same argument, it is easily verified that $J_{\theta,q}(\bar{A})$ is also a c-normed space (with the same c).

In the proof of Theorem 3.2.3, we used the triangle inequality also when we proved $J_{\theta,q}(\bar{A})\subset\Sigma(\bar{A})$. Therefore the proof of this inclusion has to be modified in the present situation. The way this modification is carried out is typical for the modifications we shall need in what follows.

In order to prove $J_{\theta,q}(\bar{A}) \subset \Sigma(\bar{A})$, we assume that $\Sigma(\bar{A})$ is c-normed and define ρ by the formula $(2c)^\rho = 2$. We can assume that c is large, so that $\rho < q$. Assuming that $a = \sum_\nu u_\nu$ in $\Sigma(\bar{A})$ with $u_\nu \in \Delta(\bar{A})$ we therefore obtain from Lemma 3.10.2

$$K(1,a) \leqslant C(\sum_\nu K(1,u_\nu)^\rho)^{1/\rho}.$$

Using the estimate

$$K(1,a) \leqslant \min(1,2^{-\nu}) J(2^\nu, a)$$

(Lemma 3.2.1), we obtain

$$K(1,a) \leqslant C(\sum_\nu 2^{\nu\rho\theta} \min(1,2^{-\rho\nu}) \cdot (2^{-\nu\theta} J(2^\nu,u_\nu))^\rho)^{1/\rho}.$$

Put $p = q/\rho$. Then $p > 1$ and therefore Hölder's inequality implies

$$K(1,a) \leqslant C(\sum_\nu 2^{\nu\rho\theta p'} \min(1,2^{-\rho\nu p'}))^{1/\rho p'} (\sum_\nu (2^{\nu\theta} J(2^\nu,u_\nu))^{\rho p})^{1/\rho p},$$

where $1/p' = 1 - 1/p$. Since $0 < \theta < 1$, the first sum on the right hand side converges. Since $\rho p = q$, we therefore obtain from Lemma 3.2.2

$$K(1,a) \leqslant C_{\theta,q} \|a\|_{\theta,q;J},$$

proving that

$$J_{\theta,q}(\bar{A}) \subset \Sigma(\bar{A}).$$

The proof of the fact that $J_{\theta,q}$ is an interpolation method of exponent θ will follow from the K-part of the theorem and the equivalence theorem below. □

3.11.3. Theorem (The equivalence theorem). *Assume that* $0 < \theta < 1$, $0 < q \leqslant \infty$, *and let* $\bar{A}$ *be a couple of quasi-normed Abelian groups. Then* $J_{\theta,q}(\bar{A}) = K_{\theta,q}(\bar{A})$ *with equivalent quasi-norms.*

Proof: First we prove $J_{\theta,q}(\bar{A}) \subset K_{\theta,q}(\bar{A})$. Take an element $a \in J_{\theta,q}(\bar{A})$, and assume $a = \sum_\nu u_\nu$. We know that $K(t,a)$ is a c-norm. Choosing c large and ρ so that $(2c)^\rho = 2$, we have $p = q/\rho > 1$, and, just as in the proof of the previous theorem,

$$K(t,a) \leqslant C(\sum_\nu (\min(1, t2^{-\nu}))^\rho J(2^\nu, u_\nu)^\rho)^{1/\rho}.$$

It follows that

$$\begin{aligned} K(2^\mu, a) &\leqslant C(\sum_\nu (\min(1,2^{\mu-\nu}) J(2^\nu, u_\nu))^\rho)^{1/\rho} \\ &= C(\sum_\nu (\min(1,2^\nu) J(2^{\mu-\nu}, u_{\mu-\nu}))^\rho)^{1/\rho}. \end{aligned}$$

Thus we obtain, by Lemma 3.1.3 and Minkowski's inequality,

$$\begin{aligned}\|a\|_{\theta,q;K} &\leqslant C(\textstyle\sum_\mu (2^{-\mu\theta}K(2^\mu,a))^q)^{1/q}\\ &= C\{(\textstyle\sum_\mu (2^{-\mu\theta\rho}(K(2^\mu,a))^\rho)^p)^{1/p}\}^{1/\rho}\\ &\leqslant C\{(\textstyle\sum_\mu [2^{-\mu\theta\rho}\sum_\nu (\min(1,2^\nu)J(2^{\mu-\nu},u_{\mu-\nu}))^\rho]^p)^{1/p}\}^{1/\rho}\\ &\leqslant C\{\textstyle\sum_\nu \min(1,2^{\nu\rho})[\sum_\mu (2^{-\mu\theta}J(2^{\mu-\nu},u_{\mu-\nu}))^q]^{1/p}\}^{1/\rho}\\ &= C\{\textstyle\sum_\nu \min(1,2^{\nu\rho})\cdot 2^{-\nu\theta\rho}\}^{1/p}\{\sum_\mu (2^{-\mu\theta}J(2^\mu,u_\mu))^q\}^{1/q}.\end{aligned}$$

By Lemma 3.2.3, we conclude that

$$\|a\|_{\theta,q;K} \leqslant C\|a\|_{\theta,q;J}.$$

The converse of this inequality can be proved just as in the case of normed spaces. In fact, the proof of the fundamental lemma goes over without any essential change. If A_j is c_j-normed we have only to change the value of the constant γ, so that γ is replaced by $\gamma\cdot\max(c_0,c_1)$. The reader is asked to check the details for himself. When the fundamental lemma is established we immediately obtain the desired result. □

It should be clear by now how to extend the results in Section 3.4. In fact, Theorem 3.4.1 needs no change, neither in its formulation nor in its proof. The proof of Theorem 3.4.2 has to be modified slightly. However it is only the proof of part (a), the completeness of $\bar{A}_{\theta,q}$, which has to be changed. We leave it to the reader to carry out this proof with the aid of Lemma 3.10.2.

The definitions of spaces of class $\mathscr{C}_K(\theta;\bar{A})$, $\mathscr{C}_J(\theta;\bar{A})$ and $\mathscr{C}(\theta;\bar{A})$ (Definition 3.5.1) carry over without change. An equivalent formulation of this definition will be given in our next theorem, which corresponds to Theorem 3.5.2.

3.11.4. Theorem. *Suppose that* $0<\theta<1$. *Then*

(a) *X is of class $\mathscr{C}_K(\theta;\bar{A})$ if and only if*

$$\Delta(\bar{A}) \subset X \subset \bar{A}_{\theta,\infty}.$$

(b) *A complete space X is of class $\mathscr{C}_J(\theta;\bar{A})$ if and only if for some $q\leqslant 1$ we have*

$$\bar{A}_{\theta,q} \subset X \subset \Sigma(\bar{A}).$$

If X is c-normed we can choose q so that $(2c)^q=2$.

Proof: Only part (b) needs a new proof. Assume that $a=\sum_\nu u_\nu$ in $\Sigma(\bar{A})$. By Lemma 3.10.2 we have, since X is complete,

$$\|a\|_X \leqslant C(\textstyle\sum_\nu \|u_\nu\|_X^q)^{1/q}.$$

If X is of class $\mathscr{C}_J(\theta;\bar{A})$ we obtain

$$\|a\|_X \leqslant C(\sum_\nu (2^{-\nu\theta} J(2^\nu, u_\nu))^q)^{1/q},$$

i.e.

$$\bar{A}_{\theta,q} \subset X.$$

The converse follows as in the proof of Theorem 3.5.2. □

3.11.5. Theorem (The reiteration theorem). *Let $\bar{A}=(A_0, A_1)$ and $\bar{X}=(X_0, X_1)$ be two couples of quasi-normed Abelian groups, and assume that X_i $(i=0,1)$ are complete spaces of class $\mathscr{C}(\theta_i;\bar{A})$, where $0\leqslant\theta_i\leqslant 1$ and $\theta_0\neq\theta_1$. Put*

$$\theta=(1-\eta)\theta_0+\eta\theta_1, \quad 0<\eta<1.$$

Then

$$\bar{X}_{\eta,q}=\bar{A}_{\theta,q}, \quad 0<q<\infty.$$

In particular, we have

$$(\bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_1,q_1})_{\eta,q}=\bar{A}_{\theta,q}, \quad 0<q<\infty.$$

Proof: The proof of the inclusion $\bar{X}_{\eta,q}\subset\bar{A}_{\theta,q}$ goes through as in the normed case (Theorem 3.5.3). The proof of the converse inclusion, however, has to be changed. Assume therefore that $a\in\bar{A}_{\theta,q}$, and choose a representation $a=\sum_\nu u_\nu$. As in the proof of Theorem 3.5.3, we change variables and then we apply Lemma 3.1.3:

$$\|a\|_{\bar{X}_{\eta,q}} \leqslant C(\sum_\mu (2^{-\mu(\theta-\theta_0)} K(2^{\mu(\theta_1-\theta_0)}, a;\bar{X}))^q)^{1/q}.$$

To estimate the right hand side, we note, again as in the proof of Theorem 3.5.3, that

$$2^{\mu\theta_0} K(2^{\mu(\theta_1-\theta_0)}, u_\nu;\bar{X}) \leqslant C 2^{(\mu-\nu)\theta_0} \min(1, 2^{(\mu-\nu)(\theta_1-\theta_0)}) J(2^\nu, u_\nu;\bar{A}),$$

and, by Lemma 3.10.2, that for any $\rho>0$, small enough,

$$K(t, a;\bar{X}) \leqslant C(\sum_\nu (K(t, u_{\mu-\nu};\bar{X}))^\rho)^{1/\rho}.$$

Using these two observations, we infer that $(p=q/\rho>1)$

$$\begin{aligned}
\|a\|_{\bar{X}_{\eta,q}} &\leqslant C(\sum_\mu 2^{-\mu\theta q}(\sum_\nu (2^{\mu\theta_0} K(2^{\mu(\theta_1-\theta_0)}, u_{\mu-\nu};\bar{X}))^\rho)^{q/\rho})^{1/q} \\
&\leqslant C\{(\sum_\mu [2^{-\mu\theta\rho}(\sum_\nu (2^{\nu\theta_0}\min(1, 2^{\nu(\theta_1-\theta_0)}) J(2^{\mu-\nu}, u_{\mu-\nu};\bar{A}))^\rho]^p)^{1/p}\}^{1/\rho} \\
&\leqslant C\{\sum_\nu 2^{\nu\theta_0\rho}\min(1, 2^{\nu(\theta_1-\theta_0)\rho})(\sum_\mu (2^{-\mu\theta} J(2^{\mu-\nu}, u_{\mu-\nu};\bar{A}))^q)^{1/p}\}^{1/\rho} \\
&\leqslant C\{\sum_\nu (2^{\nu(\theta_0-\theta)}\min(1, 2^{\nu(\theta_1-\theta_0)}))^\rho\}^{1/\rho}(\sum_\mu (2^{-\mu\theta} J(2^\mu, u_\mu;\bar{A}))^q)^{1/q},
\end{aligned}$$

by Minkowski's inequality. Taking the infimum and invoking the equivalence Theorem 3.11.3, we obtain

$$\|a\|_{\bar{X}_{\eta,q}} \leqslant C\|a\|_{\bar{A}_{\theta,q}}. \quad \square$$

Next, we draw attention to the fact that if $\|\cdot\|$ is a quasi-norm on A then so is $\|\cdot\|^\rho$ for all $\rho>0$. The proof of this is obvious. Let us denote by $(A)^\rho$ the space A provided with the quasi-norm $\|\cdot\|^\rho$. Note that a proper choice of ρ (see Lemma 3.10.1) will make $(A)^\rho$ a 1-normed space. Now it is natural to ask for a connection between the spaces $\bar{A}_{\theta,q}$ and the spaces $((A_0)^{\rho_0}, (A_1)^{\rho_1})_{\theta,q}$. Such a connection is given in our next theorem.

3.11.6. Theorem (The power theorem). *Let ρ_0 and ρ_1 be given positive numbers and put*

$$\theta = \eta\rho_1/\rho\,,$$
$$\rho = (1-\eta)\rho_0 + \eta\rho_1\,,$$
$$q = \rho r\,.$$

Then

$$((A_0)^{\rho_0}, (A_1)^{\rho_1})_{\eta,r} = ((A_0, A_1)_{\theta,q})^\rho\,,$$

where $0<\eta<1$, $0<r\leqslant\infty$.

In the proof of the power theorem we shall work with the functional (cf. Exercise 1)

$$K_\infty(t,a) = K_\infty(t,a;\bar{A}) = \inf_{a=a_0+a_1}\max(\|a_0\|_{A_0}, t\|a_1\|_{A_1})\,.$$

Since

$$K_\infty(t,a) \leqslant K(t,a) \leqslant 2K_\infty(t,a)\,,$$

the norm on $\bar{A}_{\theta,q}$ will be equivalent to $\Phi_{\theta,q}(K_\infty(t,a))$. We now have the following lemma:

3.11.7. Lemma. *Let ρ_0 and ρ_1 be given positive numbers. Then*

$$K_\infty(s,a;(A_0)^{\rho_0}, (A_1)^{\rho_1}) = (K_\infty(t,a;A_0,A_1))^{\rho_0},$$

if

$$s = t^{\rho_1}(K_\infty(t,a;A_0,A_1))^{\rho_0-\rho_1}.$$

Proof: For simplicity we write

$$K_\infty(t) = K_\infty(t,a;A_0,A_1)\,,$$
$$K_\infty(s) = K_\infty(s,a;(A_0)^{\rho_0}, (A_1)^{\rho_1})\,.$$

Now choose a_0 and a_1 so that ($\varepsilon>0$ is arbitrarily given)

$$K_\infty(t)\leqslant\max(\|a_0\|_{A_0},t\|a_1\|_{A_1})\leqslant(1+\varepsilon)K_\infty(t).$$

Since at least one of the numbers

$$\|a_0\|_{A_0}/K_\infty(t) \quad\text{and}\quad t\|a_1\|_{A_1}/K_\infty(t)$$

is larger than 1 and both are smaller than $1+\varepsilon$, we have

$$1\leqslant\max\left(\left(\frac{\|a_0\|_{A_0}}{K_\infty(t)}\right)^{\rho_0},\left(\frac{t\|a_1\|_{A_1}}{K_\infty(t)}\right)^{\rho_1}\right)\leqslant 1+\varepsilon'$$

where $\varepsilon'\to 0$ as $\varepsilon\to 0$. This implies

$$1=\inf_{a=a_0+a_1}\max\left(\left(\frac{\|a_0\|_{A_0}}{K_\infty(t)}\right)^{\rho_0},\left(\frac{t\|a_1\|_{A_1}}{K_\infty(t)}\right)^{\rho_1}\right),$$

which clearly gives the lemma. □

Proof (of the power theorem): If $q=\infty$ we have $r=\infty$, and thus, with the notation of the proof of Lemma 3.11.7,

$$\|a\|_{((A_0)^{\rho_0},(A_1)^{\rho_1})_{\eta,\infty}}\sim\sup_{s>0}s^{-\eta}K_\infty(s)=\sup_{t>0}t^{-\eta\rho_1}(K_\infty(t))^{\rho}.$$

Since $\eta\rho_1=\theta\rho$ this gives the result.

In the case $q<\infty$ we also have $r<\infty$. Noting that $K_\infty(t)$ is a decreasing function of t, we have

$$\|a\|^r_{((A_0)^{\rho_0},(A_1)^{\rho_1})_{\eta,r}}\sim\int_0^\infty(s^{-\eta}K_\infty(s))^r ds/s\sim-\int_0^\infty(K_\infty(s))^r ds^{-\eta r}$$
$$\sim\int_0^\infty s^{-\eta r}d(K_\infty(s))^r.$$

In the right hand side we change the variable s to the variable t. By Lemma 3.11.7 the right member then becomes equivalent to

$$\int_0^\infty t^{-\eta\rho_1 r}(K_\infty(t))^{-\eta r(\rho_0-\rho_1)}d(K_\infty(t))^{\rho_0 r}$$
$$\sim\int_0^\infty t^{-\theta q}d(K_\infty(t))^q\sim\int_0^\infty t^{-\theta q}(K_\infty(t))^q dt/t.$$

This proves the theorem. □

The power theorem gives at once the following interpolation result.

3.11.8. Theorem. *Suppose that* $T\colon A_i\to B_i$ *with quasi-norm* M_i $(i=0,1)$. *Then*

$$T\colon((A_0)^{\rho_0},(A_1)^{\rho_1})_{\eta,r}\to((B_0)^{\rho_0},(B_1)^{\rho_1})_{\eta,r}$$

with quasi-norm M, *such that*

$$M\leqslant M_0^{(1-\eta)\rho_0}M_1^{\eta\rho_1}.$$

3.12. Some Other Equivalent Real Interpolation Methods

In this section we describe a few other real interpolation methods which are frequently used in the literature. These methods are known as "espaces de moyennes" and "espaces de traces". We shall give a brief presentation and also prove the fact that these interpolation methods are equivalent to the K- and J-method. (The historical development is the reverse one, the K- and J-methods being more recent than the methods introduced in this section.) In order to align the notation with the rest of the book we have modified the original definitions given by Lions (cf. Notes and Comment).

"Espaces de moyennes"

For simplicity we shall work in the category $\mathscr{B}$ of all Banach spaces. If A is a Banach space we let $L_p^*(A)$ denote the space of all A-valued, strongly measurable functions u on $\mathbb{R}_+ = \{t: 0<t<\infty\}$, such that the norm

$$\|u(t)\|_{L_p^*(A)} = (\int_0^\infty \|u(t)\|_A^p \, dt/t)^{1/p}$$

is finite. Here we take $1 \leqslant p \leqslant \infty$, with the usual convention when $p=\infty$.

Let p_0 and p_1 be two numbers such that $1 \leqslant p_j \leqslant \infty$ $(j=0,1)$ and put $\bar{p}=(p_0, p_1)$. Moreover let θ $(0<\theta<1)$ be given. For a given compatible Banach couple $\bar{A}=(A_0, A_1)$ we now define the space $S=S(\bar{A}, \bar{p}, \theta)$ as follows: *S is the subspace of $\Sigma(\bar{A})$ consisting of all a for which there is a representation*

$$a = \int_0^\infty u(t) \, dt/t \quad (\textit{convergence in } \Sigma(\bar{A}))$$

where $u(t) \in \Delta(\bar{A})$, $0<t<\infty$ *and*

$$\max(\|t^{-\theta} u(t)\|_{L_{p_0}^*(A_0)}, \|t^{1-\theta} u(t)\|_{L_{p_1}^*(A_1)}) < \infty .$$

The norm on S is the infimum of the left hand side over all admissible u.

The space $S(\bar{A}, \bar{p}, \theta)$ is one of two spaces called "espaces de moyennes" in Lions-Peetre [1]. *A second space, denoted by $\underline{S} = \underline{S}(\bar{A}, \bar{p}, \theta)$, is defined by means of the norm*

$$\inf_{a=a_0(t)+a_1(t)} (\|t^\theta a_0(t)\|_{L_{p_0}^*(A_0)} + \|t^{1-\theta} a_1(t)\|_{L_{p_1}^*(A_1)}) .$$

3.12.1. Theorem. *Let $\bar{p}=(p_0, p_1)$ and θ be given, so that $1 \leqslant p_j < \infty$, $j=0,1$ and $0<\theta<1$. Put*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} .$$

Then, for any Banach couple $\bar{A}$, we have, with equivalent norms,

$$S(\bar{A}, \bar{p}, \theta) = \underline{S}(\bar{A}, \bar{p}, \theta) = \bar{A}_{\theta, p} .$$

Proof: In order to prove the first equality $S=\underline{S}$ (this is essentially the equivalence theorem), we assume that $a=\int_0^\infty u(\tau)d\tau/\tau \in S$. Put

$$a_0(t)=\int_0^1 u(t\tau)d\tau/\tau, \qquad a_1(t)=\int_1^\infty u(t\tau)d\tau/\tau.$$

Then, by Minkowski's inequality,

$$\|t^{-\theta}a_0(t)\|_{L^*_{p_0}(A_0)} \leqslant \theta^{-1}\|t^{-\theta}u(t)\|_{L^*_{p_0}(A_0)}$$

and

$$\|t^{1-\theta}a_1(t)\|_{L^*_{p_1}(A_1)} \leqslant (1-\theta)^{-1}\|t^{1-\theta}u(t)\|_{L^*_{p_1}(A_1)}$$

which clearly implies $S\subset\underline{S}$. Conversely, assume that $a=a_0(t)+a_1(t)\in\underline{S}$, where $t^{-\theta}a_0(t)\in L^*_{p_0}(A_0)$ and $t^{1-\theta}a_1(t)\in L^*_{p_1}(A_1)$. Let ρ be an infinitely differentiable function with compact support on $\mathbb{R}_+$ such that $\int_0^\infty \rho(\tau)d\tau/\tau=1$ and put

$$\tilde{a}_j(t)=\int_0^\infty \rho(t/\tau)a_j(\tau)d\tau/\tau.$$

Then $\tilde{a}_0(t)+\tilde{a}_1(t)=a$, $\tilde{a}_j(t)\to 0$ as $t\to 0$ or $t\to\infty$ and

$$\|t^{-\theta}\cdot t\tilde{a}_0'(t)\|_{L^*_{p_0}(A_0)} \leqslant C\|t^{-\theta}a_0(t)\|_{L^*_{p_0}(A_0)},$$
$$\|t^{1-\theta}\cdot t\tilde{a}_1'(t)\|_{L^*_{p_1}(A_1)} \leqslant C\|t^{1-\theta}a_1(t)\|_{L^*_{p_1}(A_1)}.$$

Writing $u(t)=t\tilde{a}_0'(t)=-t\tilde{a}_1'(t)$ we then have

$$\int_0^\infty u(t)dt/t=\int_0^1 \tilde{a}_0'(t)dt-\int_1^\infty \tilde{a}_1'(t)dt=\tilde{a}_0(1)+\tilde{a}_1(1)=a$$

and

$$\|a\|_S \leqslant \max(\|t^{-\theta}\cdot t\tilde{a}_0'(t)\|_{L^*_{p_0}(A_0)}, \|t^{1-\theta}\cdot t\tilde{a}_1'(t)\|_{L^*_{p_1}(A_1)})$$
$$\leqslant C\max(\|t^{-\theta}a_0(t)\|_{L^*_{p_0}(A_0)}, \|t^{1-\theta}a_1(t)\|_{L^*_{p_1}(A_1)}).$$

This clearly implies $\underline{S}\subset S$. Thus we have proved that $S=\underline{S}$.

Next, we prove that $\underline{S}=\bar{A}_{\theta,p}$ (this is essentially the power theorem). We shall use the fact that

$$\|a\|_{\underline{S}} \sim \inf_{a=a_0(t)+a_1(t)}(\|t^{-\theta}a_0(t)\|^{p_0}_{L^*_{p_0}(A_0)}+\|t^{1-\theta}a_1(t)\|^{p_1}_{L^*_{p_1}(A_1)})^{1/p}. \tag{1}$$

In order to prove (1) we choose a_0 and a_1 so that $a=a_0(t)+a_1(t)$ and

$$(\|t^{-\theta}a_0(t)\|^{p_0}_{L^*_{p_0}(A_0)}+\|t^{1-\theta}a_1(t)\|^{p_1}_{L^*_{p_1}(A_1)})^{1/p}<\infty.$$

If $a \neq 0$ we can assume that each of the two terms on the left hand side is positive. Thus, choosing λ appropriately, we see that

$$\begin{aligned}\|a\|_{\underline{S}} &\leqslant \max(\|t^{-\theta}a_0(\lambda t)\|_{L^*_{p_0}(A_0)}, \|t^{1-\theta}a_1(\lambda t)\|_{L^*_{p_1}(A_1)})\\ &= \max(\lambda^{\theta}\|t^{-\theta}a_0(t)\|_{L^*_{p_0}(A_0)}, \lambda^{\theta-1}\|t^{1-\theta}a_1(t)\|_{L^*_{p_1}(A_1)})\\ &= \|t^{-\theta}a_0(t)\|^{1-\theta}_{L^*_{p_0}(A_0)} \cdot \|t^{1-\theta}a_1(t)\|^{\theta}_{L^*_{p_1}(A_1)}\\ &\leqslant (\|t^{-\theta}a_0(t)\|^{p_0}_{L^*_{p_0}(A_0)} + \|t^{1-\theta}a_1(t)\|^{p_1}_{L^*_{p_1}(A_1)})^{1/p}.\end{aligned}$$

This implies half of (1). The remaining half is proved in the same way.

Using (1) we see that

$$\begin{aligned}\|a\|^p_{\underline{S}} &\sim \int_0^\infty \inf(t^{-\theta p_0}\|a_0(t)\|^{p_0}_{A_0} + t^{(1-\theta)p_1}\|a_1(t)\|^{p_1}_{A_1})\,dt/t\\ &\sim \int_0^\infty t^{-\eta}\inf(\|\tilde{a}_0(\tau)\|^{p_0}_{A_0} + \tau\|\tilde{a}_1(\tau)\|^{p_1}_{A_1})\,d\tau/\tau,\end{aligned}$$

where $\eta p_1 = \theta p$, $\tilde{a}_j(\tau) = a_j(t)$, $(j=0,1)$ and $\tau^{\eta} = t^{\theta p_0}$. Using the power theorem we therefore see that

$$(\underline{S})^p = ((A_0)^{p_0}, (A_1)^{p_1})_{\eta,1} = ((A_0, A_1)_{\theta,p})^p$$

which proves that $\underline{S} = \bar{A}_{\theta,p}$. □

"Espaces de traces"

If u is an A-valued function on $\mathbb{R}_+$ we let u' denote the derivative in the sense of distribution theory. We shall work with the space $V^m = V^m(\bar{A}, \bar{p}, \theta)$ of all functions u on $\mathbb{R}_+$ with values in $\Sigma(\bar{A})$, such that u is locally A_0-integrable, $u^{(m)}$ is locally A_1-integrable and such that

$$\|u\|_{V^m} = \max(\|t^{\theta}\cdot u(t)\|_{L^*_{p_0}(A_0)}, \|t^{\theta-1}\cdot u^{(m)}(t)\|_{L^*_{p_1}(A_1)})$$

is finite. We assume that $0<\theta<1$ and $1 \leqslant p_0 \leqslant \infty$, $1 \leqslant p_1 \leqslant \infty$. Then $\|\cdot\|_{V^m}$ is a norm, and V^m is a Banach space.

We shall say that $u(t)$ has a trace in $\Sigma(\bar{A})$ if $u(t)$ converges in $\Sigma(\bar{A})$ as $t \to 0$. Then we put

$$\operatorname{trace} u = \lim_{t\to 0} u(t).$$

The space of traces of functions in V^m will be denoted by $T^m = T^m(\bar{A}, \bar{p}, \theta)$. Thus *$T^m$ is the space of all $a \in \Sigma(\bar{A})$, such that there is a function $u \in V^m$ with* trace $u = a$. *Introducing the quotient norm*

$$\|a\|_{T^m} = \inf_{\operatorname{trace} u = a}\|u\|_{V^m},$$

T becomes a Banach space.

3.12.2. Theorem. *Assume that* $0<\theta<1$ *and* $1\leqslant p_j<\infty$ *for* $j=0,1$. *Then*

$$T^m(\bar{A},\bar{p},\theta)=\bar{A}_{\theta,p},$$

where

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

Proof: We shall prove that

$$(2)\qquad T^m(\bar{A},\bar{p},\theta)=S(\bar{A},\bar{p},\theta)$$

which gives the result in view of Theorem 3.12.1.

First assume that $a\in T^m$. Then $a=\operatorname{trace} u$ for some $u\in V^m$. Let φ be an infinitely differentiable function with compact support on $\mathbb{R}_+$ such that

$$\int_0^\infty \varphi(t)\,dt/t=1\,.$$

Put

$$\tilde{u}(t)=\int_0^\infty \varphi(t/\tau)\,u(\tau)\,d\tau/\tau\,.$$

Then it is easily seen that

$$a=\int_0^\infty v(t)\,dt/t=\int_0^\infty v(1/t)\,dt/t\,,$$

if for some constant d

$$v(t)=d\cdot t^m\tilde{u}^{(m)}(t)\,.$$

Clearly

$$v(t)=d\int_0^\infty (t/\tau)^m\varphi^{(m)}(t/\tau)\,u(\tau)\,d\tau/\tau\,,$$

and thus

$$\|t^{-\theta}v(1/t)\|_{L^*_{p_0}(A_0)}=\|t^\theta v(t)\|_{L^*_{p_0}(A_0)}\leqslant C\,\|t^\theta u(t)\|_{L^*_{p_0}(A_0)}\,.$$

Moreover, since

$$v(t)=d\int_0^\infty \varphi(t/\tau)\,\tau^m u^{(m)}(\tau)\,d\tau/\tau$$

we also have

$$\|t^{1-\theta}v(1/t)\|_{L^*_{p_1}(A_1)}\leqslant C\,\|t^{\theta-1}\cdot t^m u^{(m)}(t)\|_{L^*_{p_1}(A_1)}\,.$$

It follows that $\|a\|_S\leqslant C\,\|u\|_{V^m}$ and hence $T^m\subset S$.

Conversely, assume that $a\in S$ has the representation

$$a=\int_0^\infty v(t)\,dt/t\,.$$

Then we put

$$u(t)=\int_t^\infty (1-t/\tau)^{m-1}\, v(1/\tau)\, d\tau/\tau\,.$$

Clearly $\operatorname{trace} u=a$ and, by Minkowski's inequality,

$$\|t^\theta u(t)\|_{L^*_{p_0}(A_0)}\leqslant c_1\,\|t^\theta v(1/t)\|_{L^*_{p_0}(A_0)}=c_1\,\|t^{-\theta}v(t)\|_{L^*_{p_0}(A_0)}\,.$$

Moreover

$$(-1)^m t^m u^{(m)}(t)=v(1/t)$$

so that

$$\|t^{\theta-1}t^m u^{(m)}(t)\|_{L^*_{p_1}(A_1)}=\|t^{1-\theta}v(t)\|_{L^*_{p_1}(A_1)}\,.$$

It follows that $a\in T^m$ and thus $S\subset T^m$. □

In Section 6.6 we shall need a modified equivalent definition of the space T^m. Let $\bar\eta=(\eta_0,\eta_1)$ be given and define the space $\tilde V^m=\tilde V^m(\bar A,\bar p,\bar\eta)$ by means of the norm

$$\|u\|_{\tilde V^m}=\max(\|t^{\eta_0}u(t)\|_{L^*_{p_0}(A_0)},\ \|t^{\eta_1}u^{(m)}(t)\|_{L^*_{p_1}(A_1)})\,.$$

Then we have

3.12.3. Corollary. *Assume that* $\eta_0>0$, $\eta_1<m$ *and* $1\leqslant p_j<\infty$ *for* $j=0,1$. *Put*

$$\theta=\eta_0/(\eta_0+m-\eta_1),\qquad 1/p=(1-\theta)/p_0+\theta/p_1\,.$$

Then

$$\|a\|_{\bar A_{\theta,p}}\sim\inf_{\operatorname{trace} u=a}\|u\|_{\tilde V^m}\,.$$

Proof: First we note that

$$\|u\|_{V^m}\sim\max(\|t^\theta u(t)\|_{L^*_{p_0}(A_0)},\ \|t^{\theta-1}(td/dt)^m u(t)\|_{L^*_{p_1}(A_1)})\,. \tag{3}$$

In order to see this we observe that $(td/dt)^m u(t)$ is a linear combination of $t^k u^{(k)}(t)$, $k=1,\ldots,m$. Moreover

$$t^k u^{(k)}(t)=c_{m,k}\int_t^\infty (t/\tau)^k(1-t/\tau)^{m-k-1}\tau^m u^{(m)}(\tau)\,d\tau/\tau$$

so that

$$\|t^{\theta-1}\cdot t^k u^{(k)}(t)\|_{L^*_{p_1}(A_1)}\leqslant C\,\|t^{\theta-1}t^m u^{(m)}(t)\|_{L^*_{p_1}(A_1)}\,.$$

Thus the right hand side of (3) is bounded by a constant multiplied by the norm of u in V^m.

Conversely, we obviously have

$$(td/dt)\,u(t)=-\int_t^\infty(\tau d/d\tau)^2u(\tau)\,d\tau/\tau$$

and therefore, by Minkowski's inequality,

$$\|t^{\theta-1}(td/dt)\,u(t)\|_{L^*_{p_1}(A_1)} \leqslant C\,\|t^{\theta-1}(td/dt)^2 u(t)\|_{L^*_{p_1}(A_1)}\,.$$

Writing $t^m u^{(m)}$ as a linear combination of $(td/dt)^k u$ $(k=1,\dots,m)$, this estimate clearly completes the proof of (3).

Now we change the variable of integration on the right hand side of (3), writing $u(t)=v(s)$, if $t=s^\rho$. Then

$$(td/dt)^m u(t) = c(sd/ds)^m v(s)\,.$$

It follows that

$$\begin{aligned}\|u\|_{V^m} &\sim \max(\|s^{\theta\rho}v(s)\|_{L^*_{p_0}(A_0)}, \|s^{(\theta-1)\rho}(sd/ds)^m v(s)\|_{L^*_{p_1}(A_1)})\\ &\sim \max(\|s^{\theta\rho}v(s)\|_{L^*_{p_0}(A_0)}, \|s^{(\theta-1)\rho+m}v^{(m)}(s)\|_{L^*_{p_1}(A_1)})\,.\end{aligned}$$

With $\rho=\eta_0+m-\eta_1$ we finally see that

$$\|u\|_{V^m} \sim \|v\|_{\tilde{V}^m}\,.$$

Since $\operatorname{trace} u = \operatorname{trace} v$ (ρ being positive) we get the result of the corollary. □

3.13. Exercises

1. (Holmstedt-Peetre [1]). Let $\bar{A}$ be a couple of quasi-normed spaces. Define the functional $K_p(t,a)$ by

$$K_p(t) = K_p(t,a) = \inf_{a=a_0+a_1}(\|a_0\|^p_{A_0} + t^p\|a_1\|^p_{A_1})^{1/p}\,.$$

Show that $\Phi_{\theta,q}(K_p(t,a))$ is an equivalent quasi-norm on $\bar{A}_{\theta,q}$ for all $p>0$. Prove, moreover, that

$$K_p(t) = \inf_s(1+(t/s)^r)^{1/r} K_q(s)\,,$$

where $1\leqslant p\leqslant q\leqslant\infty$ and $1/r=1/p-1/q$, and that

$$K_p(t) = \sup_s(1+(t/s)^r)^{-1/r} K_q(s)\,,$$

where $1\leqslant q\leqslant p\leqslant\infty$ and $1/r=1/q-1/p$.

Hint: Use Hölder's inequality and the Gagliardo diagram.

2. (Holmstedt-Peetre [1]). Prove that (in the notation of the previous exercise)

$$K_\infty(t,b) \leqslant K_\infty(t,a) \qquad (t>0)$$

iff for each $\varepsilon>0$ and each decomposition $a_0+a_1=a$ of a there is a decomposition $b_0+b_1=b$ of b such that $\|b_i\|_{A_i} \leqslant \|a_i\|_{A_i}+\varepsilon$.

3. (Holmstedt-Peetre [1]). With the notation of Exercise 1, show that if

$$K_p(t,b) \leqslant K_p(t,a) \qquad (t>0)$$

holds for some $p \geqslant 1$, then it holds for all $p \geqslant 1$. Cf. Sparr [2].

Hint: Apply Exercise 1 and Exercise 2.

4. Prove that under suitable conditions on the spaces involved we have

$$(A_0^{(1)} \times A_0^{(2)}, A_1^{(1)} \times A_1^{(2)})_{\theta,q} = (A_0^{(1)}, A_1^{(1)})_{\theta,q} \times (A_0^{(2)}, A_1^{(2)})_{\theta,q}.$$

5. (a) (Lions-Peetre [1]). Let $\bar{A}^{(\nu)}$, $\nu=1,2$ and $\bar{B}$ be compatible Banach couples. Assume that T is a bilinear mapping from the couple $(A_0^{(1)} \times A_0^{(2)}, A_1^{(1)} \times A_1^{(2)})$ to $\bar{B}$ and that

$$\|T(a^{(1)}, a^{(2)})\|_{B_0} \leqslant M_0 \|a^{(1)}\|_{A_0^{(1)}} \|a^{(2)}\|_{A_0^{(2)}},$$

$$\|T(a^{(1)}, a^{(2)})\|_{B_1} \leqslant M_1 \|a^{(1)}\|_{A_1^{(1)}} \|a^{(2)}\|_{A_1^{(2)}}.$$

Prove that

$$T: \bar{A}_{\theta,p_1}^{(1)} \times \bar{A}_{\theta,p_2}^{(2)} \to \bar{B}_{\theta,q}$$

if $0<\theta<1$, $1/q-1=\sum_{\nu=1}^2 (1/p_\nu - 1)$, $1 \leqslant q \leqslant \infty$. Generalize to multilinear mappings and quasi-norms.

Hint: Apply Young's inequality.

(b) Assume that T is bilinear and that, as in (a),

$$T: \begin{cases} A_0 \times B_0 \to C_0 \\ A_0 \times B_1 \to C_1, \\ A_1 \times B_0 \to C_1 \end{cases}$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are compatible Banach couples. Show that $(h=T(f,g))$

$$K(t,h) \leqslant C\|f\|_{A_0} K(t,g) \qquad (f \in A_0, g \in \Sigma(\bar{B}))$$

$$K(t,h) \leqslant C\, K(t,f)\|g\|_{B_0} \qquad (f \in \Sigma(\bar{A}), g \in B_0).$$

Use this to prove that $(f \in \Sigma(\bar{A}), g \in \Sigma(\bar{B}))$

$$K(t,h) \leqslant C \int_1^\infty s^{-1} K(st,f) K(st,g)\, ds/s,$$

and then, if $1 \leqslant 1/p + 1/q$ and $\theta = \theta_0 + \theta_1$, that

$$T\colon \bar{A}_{\theta_0, pr} \times \bar{B}_{\theta_1, qr} \to \bar{C}_{\theta, r} \qquad (0 < \theta_i,\ \theta < 1,\ 1 \leqslant p, q, r \leqslant \infty).$$

(Cf. O'Neil [1].)

In the next two exercises we introduce and apply the concept *quasi-linearizable couples.*

6. (Peetre [10]). Let $\bar{A}$ be a compatible Banach couple. Assume that there are two families of operators $V_0(t)$ and $V_1(t)$, both in $L(\Sigma(\bar{A}), \Delta(\bar{A}))$, such that there is a number $k \geqslant 1$ for which

$$\begin{aligned} &V_0(t) + V_1(t) = I \quad \textit{(identity)}, \\ &\|V_0(t)a\|_{A_0} \leqslant k \min(\|a\|_{A_0}, t\|a\|_{A_1}), \\ &t\|V_1(t)a\|_{A_1} \leqslant k \min(\|a\|_{A_0}, t\|a\|_{A_1}). \end{aligned} \qquad (a \in \Delta(\bar{A}))$$

A couple $\bar{A}$ with these properties will be called *quasi-linearizable.* Show that for such couples

$$K(t, a) \leqslant \|V_0(t)a\|_{A_0} + t\|V_1(t)a\|_{A_1} \leqslant 2k K(t, a).$$

If $\bar{A}$ is quasi-linearizable, $\bar{B}$ is any compatible Banach couple and $P\colon \bar{B} \to \bar{A}$, $Q\colon \bar{A} \to \bar{B}$ are both linear and bounded with $QP = I$, then prove that $\bar{B}$ is also quasi-linearizable. See Exercise 18.

7. Let $A_0, A_1^{(1)}$ and $A_1^{(2)}$ be Banach spaces with $A_1^{(j)} \subset A_0$ for $j = 1, 2$. Assume that $(A_0, A_1^{(1)})$ and $(A_0, A_1^{(2)})$ are quasi-linearizable couples and let $(V_0^{(1)}(t), V_1^{(1)}(t))$ and $(V_0^{(2)}(t), V_1^{(2)}(t))$ be the corresponding couples of operators (see Exercise 6). Prove that if the operators $V_0^{(1)}(t)$ and $V_0^{(2)}(t)$ commute and

$$\|V_1^{(j)}(t)a\|_{A_1^{(k)}} \leqslant C\|a\|_{A_1^{(k)}}, \qquad j, k = 1, 2,$$

then $(A_0, A_1^{(1)} \cap A_1^{(2)})$ is a quasi-linearizable couple and

$$(A_0, A_1^{(1)} \cap A_1^{(2)})_{\theta, q} = (A_0, A_1^{(1)})_{\theta, q} \cap (A_0, A_1^{(2)})_{\theta, q}.$$

(See Notes and Comment.)

8. (Peetre [29]). Let $\bar{A}$ and $\bar{B}$ be compatible Banach couples. Prove that $(0 < \theta < 1)$

(i) $T \in L(A_0, B_0) \cap L(A_1, B_1) \Rightarrow T \in L(\bar{A}_{\theta, p}, \bar{B}_{\theta, p}) \quad (0 < p \leqslant \infty)$,

(ii) $T \in (L(A_0, B_0), L(A_1, B_1))_{\theta, 1} \Rightarrow T \in L(\bar{A}_{\theta, p}, \bar{B}_{\theta, p}) \quad (1 \leqslant p \leqslant \infty)$,

(iii) $T \in (L(A_0, B_0), L(A_1, B_1))_{\theta, \infty} \Rightarrow T \in L(\bar{A}_{\theta, 1}, \bar{B}_{\theta, \infty})$.

9. (Tartar [1]). Let $\bar{A}$ and $\bar{B}$ be compatible couples of normed linear spaces and assume that $A_1 \subset A_0$. Let T be a *non-linear mapping*, which maps A_0 to B_0 and A_1 to B_1 and suppose that there are positive increasing functions f and g and positive numbers α_0 and α_1 such that

$$\|Ta - Ta_1\|_{B_0} \leqslant f(\max(\|a\|_{A_0}, \|a_1\|_{A_0}))\,\|a-a_1\|_{A_0}^{\alpha_0},$$
$$\|Ta_1\|_{B_1} \leqslant g(\|a_1\|_{A_0})\,\|a_1\|_{A_1}^{\alpha_1}.$$

Show that T maps $A = \bar{A}_{\eta,r}$ to $B = \bar{B}_{\theta,q}$, where $\eta = \theta\alpha_1/\alpha$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, and $r = \alpha q$, and that there is a positive increasing function h, such that

$$\|Ta\|_B \leqslant h(\|a\|_{A_0})\,\|a\|_A.$$

Hint: Use the power theorem to reduce the proof to the case $\alpha_0 = \alpha_1 = 1$.

10. (Tartar [1]). Let $\bar{A}$ and $\bar{B}$ be compatible normed linear couples and assume that $A_1 \subset A_0$. Let U be an open set in A_0 and let T be an non-linear mapping from U to B_0 and from $U \cap A_1$ to B_1. Moreover, assume that, for all $a \in U$, there is a neighbourhood V in A_0 of a, such that

$$\|Ta - Ta_1\|_{B_0} \leqslant \sigma\|a-a_1\|_{A_0}^{\alpha_0},$$
$$\|Ta_1\|_{B_1} \leqslant \gamma(\|a_1\|_{A_1}^{\alpha_1} + 1),$$

where $a_1 \in V \cap A_1$ and σ, γ are constants depending on V only. Let A and B be the spaces defined in the previous exercise. Prove that T maps $U \cap A$ to B.

11. (Peetre [10]). Let Φ be a functional defined on positive Lebesgue-measurable functions f on $(0,\infty)$. We say that *Φ is a function norm* if

(1) $\Phi(f) \geqslant 0$ *for all* f,

(2) $\Phi(f) = 0 \Leftrightarrow f = 0$ *(a.e.)*,

(3) $\Phi(f) < \infty \Rightarrow f < \infty$ *(a.e.)*,

(4) $\Phi(\lambda f) = \lambda\Phi(f)$ for $\lambda > 0$,

(5) $f(t) \leqslant \sum_{j=1}^{\infty} f_j(t) \Rightarrow \Phi(f) \leqslant \sum_{j=1}^{\infty} \Phi(f_j)$.

Define the spaces $K_\Phi(\bar{A})$ and $J_\Phi(\bar{A})$ as in Sections 3.1 and 3.2 by replacing $\Phi_{\theta,q}$ by a general function norm Φ. Prove the analogues of Theorems 3.1.2 and 3.2.3 if Φ satisfies the additional conditions

(6) $\Phi(\min(1,t) < \infty$,

(7) $\int_0^\infty \min(1, t^{-1}) f(t)\,dt/t \leqslant C\Phi(f)$ for all f,

(8) $\Phi(f(\lambda t)) \leqslant \theta(\lambda)\,\Phi(f(t))$ for all f,

where θ is finite on $(0, \infty)$. Prove the analogue of the equivalence theorem if θ satisfies the condition

$$\int_0^\infty \theta(\lambda) \min(1, 1/\lambda)\, d\lambda/\lambda < \infty .$$

(There is an analogue of the reiteration theorem too. Moreover, it is sufficient to require $\theta(\lambda) = o(\max(1, \lambda))$ as $\lambda \to 0, \infty$.)

12. (Löfström [4]). Let $\bar{A} = (A_0, A_1)$ be a compatible Banach couple. Assume that A_0 and A_1 are Banach algebras with common multiplication. Prove that $\bar{A}_{\theta,1}$ is also a Banach algebra. Conversely, prove that if for any given couple $\bar{A}$ the space $\bar{A}_{\theta,q}$ is a Banach algebra under the given multiplication then $q=1$. (Cf. Chapter 4, Exercise 1.)

Hint: Use the couple (l_1, l_1^1) as a test couple and apply the discrete J-method.

13. (Peetre [20]). Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be compatible couples of normed linear spaces. We say that $\bar{A}$ is a (K-) subcouple of $\bar{B}$ if, for $i = 0, 1$,

$$A_i \subset B_i \quad \text{and} \quad \|a\|_{A_i} = \|a\|_{B_i}, \qquad a \in A_i ,$$

and if

$$K(t, a; \bar{A}) = K(t, a; \bar{B}), \qquad a \in \Sigma(\bar{A}) .$$

Prove that if $\Delta(\bar{A})$ is dense in A_0 and in A_1, then A is isometrically isomorphic to a subcouple of the couple $\bar{l}_\infty(\bar{\omega}) = (l_\infty(M; \omega_0, A_0), l_\infty(M; \omega_1, A_1))$ for suitable M, ω_0 and ω_1. Here $l_\infty(M; \omega, A)$ is the space of all functions f from M to A such that

$$\sup_{m \in M} \|f(m)\|_A \, \omega(m) < \infty .$$

(See Notes and Comment.)

Hint: Let M be the unit ball of $\Delta(\bar{A}')$ and put $\omega_i(m) = \|m\|_{A_i'}^{-1}$, $i = 0, 1$. The isomorphism is $a \to f_a$ where $f_a(a') = \langle a', a \rangle$. Note that

$$K_\infty(t, a; \bar{A}) = \sup_{a' \neq 0} \frac{|\langle a', a \rangle|}{\|a'\|_{A_0'} + t^{-1} \|a'\|_{A_1'}}$$

and prove that $K_\infty(t, a; \bar{A}) = K_\infty(t, f_a; \bar{l}_\infty(\bar{\omega}))$. Finally, use Exercise 1.

14. Define the space $T_j^m = T_j^m(\bar{A}, \bar{p}, \bar{\eta})$ by means of the quotient norm

$$\|a\|_{T_j^m} = \inf_{\operatorname{trace} u^{(j)} = a} \|u\|_{\tilde{V}^m} ,$$

where $\tilde{V}^m = \tilde{V}^m(\bar{A}, \bar{p}, \bar{\eta})$ is as defined in Section 3.12. Prove that

$$T_j^m(\bar{A}, \bar{p}, \bar{\eta}) = \bar{A}_{\theta, p}$$

if $0 \leqslant j < m$, $\eta_0 > 0$, $\eta_1 < m - j$ and

$$\theta = (\eta_0 + j)/(\eta_0 + m - \eta_1), \qquad 1/p = (1-\theta)/p_0 + \theta/p_1$$

and $1 \leqslant p_0 < \infty$, $1 \leqslant p_1 < \infty$. (See Lions-Peetre [1].)

15. (Holmstedt [1]). Let φ be a continuous positive function such that $u^{-1}\varphi(u)$ is decreasing and $\varphi(u)$ is increasing. Suppose that $0<\theta<1$, $0<\rho<1$, $0<p\leqslant\infty$ and $0<q\leqslant\infty$. Put

$$I_0(\varphi) = (\textstyle\int_0^\infty (\int_0^1 (t^{-\theta}s^{-\rho}\varphi(st))^p ds/s)^{q/p} dt/t)^{1/p},$$
$$I_1(\varphi) = (\textstyle\int_0^\infty (\int_1^\infty (t^{-\theta}s^{-\rho}\varphi(st))^p ds/s)^{q/p} dt/t)^{1/p}.$$

(i) Prove that there is a constant c, independent of θ, ρ, p and q, such that if

$$\alpha_0 = \frac{q(1-\rho)}{\theta-\rho}, \quad \textit{when} \quad 0<\rho<\theta,$$

$$\alpha_1 = \frac{q\rho}{\rho-\theta}, \quad \textit{when} \quad \theta<\rho<1,$$

and if $m = \min(0, 1-q/p)$, $n = \max(0, 1-q/p)$, then

$$c^{-1}\alpha_j^m \Phi_{\theta,q}(\varphi) \leqslant I_j(\varphi) \leqslant c\alpha_j^n \Phi_{\theta,q}(\varphi), \qquad j=0,1.$$

(ii) Prove the following sharp form of the reiteration theorem:

Put $X_j = \bar{A}_{\theta_j, q_j}$ *when* $0<\theta_j<1$, $\theta_0 \neq \theta_1$ *and* $0<q_j\leqslant\infty$. *Then there is a constant c which does not depend on* η *such that if* $\theta = (1-\eta)\theta_0 + \eta\theta_1$, $0<\eta<1$, *then*

$$c^{-1}\eta^{-m_0}(1-\eta)^{-m_1}\|a\|_{\bar{A}_{\theta,q}} \leqslant \|a\|_{\bar{X}_{\eta,q}} \leqslant c\eta^{n_0}(1-\eta)^{n_1}\|a\|_{\bar{A}_{\theta,q}}$$

where $m_j = \min(1/q, 1/q_j)$, $n_j = \max(1/q, 1/q_j)$, $j=0,1$.

Hint: The proof of the first part is by no means trivial although it depends only on Minkowski's inequality. For the second part use Holmstedt's formula, Section 3.5.

16. (Sagher [1]). The real interpolation method can be extended to quasi-normed Abelian semi-groups A having a zero element. Thus assume only that $\|a\|_A \geqslant 0$ with equality iff $a=0$, and that $\|a+b\|_A \leqslant c(\|a\|_A + \|b\|_A)$. Let A_0 and A_1 be two quasi-normed Abelian semi-groups, and let A be a topological semi-group and assume that A_0 and A_1 are sub-semi-groups of A and

$$\|a_n\|_{A_i} \to 0 \Rightarrow a_n \to 0 \quad \text{in} \quad A, \qquad (i=0,1).$$

Then define $\bar{A}_{\theta,q}$ in the obvious way by means of the K-functional.

(a) Check that the following interpolation theorem holds:

If $T: \bar{A} \to \bar{B}$ is a quasi-linear mapping, i.e. if there are numbers M_0 and M_1, such that, for all $a_i \in A_i$, we can find $b_i \in B_i$, such that $T(a_0+a_1)=b_0+b_1$ and $\|Ta_i\|_{A_i} \leqslant M_i \|b_i\|_{B_i}$, then $T: \bar{A}_{\theta,q} \to \bar{B}_{\theta,q}$ and

$$\|Ta\|_{\bar{B}_{\theta,q}} \leqslant M_0^{1-\theta} M_1^{\theta} \|a\|_{\bar{A}_{\theta,q}}.$$

(b) Check that the power theorem holds in this more general setting.

(c) Use the Holmstedt's formula to prove the reiteration formula

$$(\bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_1,q_1})_{\eta,q} = \bar{A}_{\theta,q}, \qquad \theta=(1-\eta)\theta_0+\eta\theta_1 .$$

(See Chapter 5, Exercise 8, for an application.)

17. (Peetre [26]). For a given quasi-Banach space E with norm $\|\cdot\|$, put

$$\|a\|^{\#} = \inf\{\textstyle\sum_{\nu=1}^{n} \|x_\nu\| : a = \sum_{\nu=1}^{n} x_\nu\} .$$

Let N be the space of all $a \in E$, such that $\|a\|^{\#}=0$ and let $E^{\#}$ be the completion of E/N in the norm induced by the semi-norm $\|\cdot\|$.

(a) Prove that $(E^{\#})' = E'$.

(b) Prove that if $\bar{A}$ is a Banach couple, such that $\Delta(\bar{A})$ is dense in A_0 and in A_1, then $(\bar{A}_{\theta q})^{\#} = \bar{A}_{\theta 1}$ for $0<q<1$, $0<\theta<1$. (See Chapter 5, Exercise 9, for an application.)

18. In a given category $\mathscr{C}$ an object A is called a *retract* of an object B if there are morphisms $I: A \to B$ and $P: B \to A$ such that PI is the identity.

(a) Prove that if $\bar{A}$ is a retract of $\bar{B}$ in the category of all (quasi-)normed spaces then $\bar{A}_{\theta q}$ is a retract of $\bar{B}_{\theta q}$ with "the same" mappings I and P.

(b) Prove that if A is a retract of B then $l_q^{s_0}(A)$ is a retract of $l_q^{s_1}(B)$, s_0 and s_1 arbitrary. (Cf. 5.6.)

(c) Prove that $L_p(w_0)$ is a retract of $L_p(w_1)$, w_i positive. (Cf. 5.4.)

19. Let A be a dense sub-space of a Hilbert space H. Identifying H with its dual, we then have $A \subset H \subset A'$. Show that $(A, A')_{1/2,2} = H$. (Cf. Chapter 2, Exercise 14.)

20. (Cwikel [3]). Let A and B be uniform interpolation spaces with respect to $\bar{A}$ and $\bar{B}$ in $\mathscr{N}_1$. Show that the condition

$$K(t, b; \bar{B}) \leqslant \omega(t) K(t, a; \bar{A}) \quad (\text{all } t>0),$$

where $\omega(t) \sim \omega(2t)$, $\int_0^\infty \omega(t)\,dt/t < \infty$ and $a \in A$, implies that $b \in B$ and

$$\|b\|_B \leqslant C\|a\|_A.$$

Hint: Use the fundamental lemma, and apply Theorem 1.6.1.

21. Let $\bar{A}$ be a compatible quasi-Banach couple. Prove that if $\bar{A}_{\theta,p} = A_0$ for some $\theta > 0$ and some $p \geqslant 1$ (>0) then $A_0 = A_1$. Generalize this to the case $\bar{A}_{\theta_0,p_0} = \bar{A}_{\theta_1,p_1}$ for some θ_i and p_i with $\theta_0 \neq \theta_1$, and $p_i \geqslant 1$ (>0). (Cf. 3.14 and Chapter 4, Exercise 4.) Does the conclusion still hold under the assumption $\bar{A}_{\theta,p_0} = \bar{A}_{\theta,p_1}$, where $0 < \theta < 1$, $p_0 \neq p_1$?

22. Show that L_0 is not discrete. (See 3.10 for the definition.)

3.14. Notes and Comment

The study of interpolation with respect to couples of Hilbert (Banach, etc.) spaces was motivated by questions connected with partial differential equations. Applications of the real method to interpolation of L_p-spaces are given in Chapter 5 and of Sobolev and Besov spaces in Chapter 6. The development of the real interpolation method stems from Lions [1] in 1958, and from Lions-Peetre [1], where the theory is developed for the first time. In the form given in this book, including the results, the real method was introduced by Peetre [10] in 1963. A preview of the real method may be seen in the proof of the Marcinkiewicz theorem (See. 1.7 for references.)

Several authors have done related work. See, e.g., Gagliardo [1], [2], Oklander [1], Krein [1], Krein-Petunin [1] (a survey), Aronszajn [1], Calderón [3], Lions-Magenes [1].

The methods of Lions [1] (espaces de traces) and Lions-Peetre [1] (espaces de moyennes) are equivalent to the K-method. This is discussed in 3.12. Gagliardo's [1], [2] method yields the same spaces (equivalent norms) as the K-method. (See, e.g., Peetre [10] and Holmstedt [1].) Oklander's [1] method is precisely the K-method, and was found independently. Krein's [1] notion, scales of spaces, may be described in the following way. Let A_α ($0 \leqslant \alpha \leqslant 1$) be a family of Banach spaces with dense inclusion, $A_\beta \subset A_\alpha$ if $\alpha < \beta$. The family (A_α) is called a *scale* if, given $0 \leqslant \alpha_0 \leqslant \alpha \leqslant \alpha_1 \leqslant 1$, A_α is of class $C_J(\theta, (A_{\alpha_0}, A_{\alpha_1}))$, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. (Cf. 3.5.) Their *minimal scale* is $\bar{A}_{\alpha,\infty}$, and their *maximal scale* is $\bar{A}_{\alpha,1}$ (equivalent norms). (Cf. 3.9.)

Interpolation of Lipschitz and Hölder operators (cf. Exercise 9 and 10) has been discussed by Peetre [17], who also considered the possibility of interpolation of metric spaces. For the metric case, see Gustavsson [2]. Many references are found in Peetre [17]. See also Tartar [1].

Extensions of the real method to *interpolation of more than two spaces* have been given (similarly) by Sparr [1], Yoshikawa [1], Kerzman [1], and Fernandez

[1]. Earlier (1966) M. Cotlar raised, in a personal communication, the question whether an extension to general cones is possible. Sparr's [1] work is an instance of a generalization of this kind. An *extension of the real method to the case of locally convex topological spaces* is found in Goulaouic [1]. *The case of quasi-normed Abelian groups*, treated in 3.11, was first considered by Kreé [1] (L_p with $0<p<1$), Holmstedt [1] (quasi-normed linear spaces), and Peetre-Sparr [1] and Sagher [1] (the general case).

Instead of the functional $\Phi_{\theta q}$, *more general functionals* may be used (see Peetre [10] and Exercise 11).

Moreover, instead of a couple (A_0, A_1), it possible to utilize two *pseudo-norms* $P_0(t, a)$ and $P_1(t, a)$ defined on some Hausdorff topological vector space A. P_0 and P_1 are then used to define functionals, analoguous to the K- and the J-functionals, denoted by M and N respectively. This generalization was proposed by Peetre [1] (see also Yoshinaga [1]).

Interpolation of semi-normed spaces has been treated by Gustavsson [1]. In particular, he shows that the equivalence theorem holds in this case too, with the obvious definitions of the K- and J-method.

Let F be an interpolation functor, and consider the couples $\bar{A}^{(1)}=(A_0, A_1^{(1)})$ and $\bar{A}^{(2)}=(A_0, A_1^{(2)})$. Put $\bar{A}=(A_0, A_1^{(1)}\cap A_2^{(2)})$. Peetre [27] has considered the question: *when is it true that*

$$F(\bar{A})=F(\bar{A}^{(1)})\cap F(\bar{A}^{(2)})?$$

The answer is it is true when, for instance, $\bar{A}^{(1)}$ and $\bar{A}^{(2)}$ are quasi-linearizable (Exercise 6), $F=K_{\theta q}$ and a certain commutativity condition is fulfilled: $A_1^{(1)}$ and $A_1^{(2)}$ are the domains of the commuting operators Λ_1 and Λ_2 acting in A_0, with a supplementary assumption on Λ_1 and Λ_2 (cf. Exercise 7 and 6.9). Triebel [4] has given an example of a couple for which equality does not hold when $F=K_{\theta q}$, as an answer to a question posed by Peetre. For results and applications, see Peetre [27] and the references given there.

There is an obvious question (first considered by Mitjagin [1] and Calderón [3], cf. 5.8): *Is it possible to obtain "all" interpolation spaces by some K-method?* For certain couples, the answer is "yes" (cf. 5.8). A precise formulation of the question is the following: Let $\bar{A}$ be any given couple and A any interpolation space with respect to $\bar{A}$. Is it true that

$$K(t, b; \bar{A})\leqslant K(t, a; \bar{A}), \qquad a\in A,$$

implies that $b\in A$ and

$$\|b\|_A\leqslant C\|a\|_A?$$

The answer is, in general, "no", as an example by Sedaev-Semenov [1] shows (see Exercise 5.7.14). Peetre [20] has given a contribution to the problem: For which couples $\bar{A}$ and $\bar{B}$ is it true that $(a\in\Sigma(\bar{A}), b\in\Sigma(\bar{B}))$

$$K(t, b; \bar{B})\leqslant K(t, a; \bar{A})$$

implies that $T \in L(\bar{A}, \bar{B})$ exists, such that $b = Ta$? He employs the result in Exercise 13 and the concept retract of Exercise 18. Clearly, when $\bar{B} = \bar{A}$, any compatible Banach couple $\bar{A}$, for which the answer is yes to this question, also yields yes to the first question, in view of Theorem 2.4.2. Couples for which the answer is yes to the first question are called *K-monotonic.*

Recently, M. Cwikel in a personal communication, has shown that, for any compatible Banach couple $\bar{A}$, the couple $(\bar{A}_{\theta_0, p_0}, \bar{A}_{\theta_1, p_1})$ is K-monotonic, provided that $0 < \theta_i < 1$, $1 \leqslant p_i \leqslant \infty$ $(i = 0, 1)$. Cwikel applies the retract methods introduced by Peetre [20] and Sparr's [2] result for weighted L_p-spaces mentioned below in 5.8. (Cf. Exercise 18 and 13.) In particular, the couple $(B^{s_0}_{p_0 q_0}, B^{s_1}_{p_1 q_1})$ of Besov spaces (see Chapter 6) is K-monotonic $(s_i \in \mathbb{R}, 1 \leqslant p_i, q_i \leqslant \infty)$, as well as the couple $(L_{p_0 q_0}, L_{p_1 q_1})$ $(1 \leqslant p_i, q_i < \infty, q_i \leqslant p_i$ $(i = 0, 1))$ (cf. Exercise 1.6.6).

3.14.1—2. As we remarked in Chapter 2, we use categories and functors only in order to obtain greater precision of expression.

The discrete versions of the K- and the J-method are frequently used in the applications, see, e.g., Chapter 6. They are also convenient for the extension of the real methods to the quasi-normed case in 3.11.

3.14.3. The fundamental lemma, employed when proving that $K_{\theta q}$ and $J_{\theta q}$ are equivalent, exhibits a universal constant. The least value of this constant is unknown to us. Peetre (unpublished) has shown that it is at least 1/2.

3.14.4. The inclusion (b) in Theorem 3.4.2 reflects a general inequality, in a way a converse to Hölder's inequality:

Let f be a positive and quasi-concave function on $\mathbb{R}_+$, i.e.

$$f(s) \leqslant \max(1, s/t) f(t).$$

Assume that $0 < p \leqslant q \leqslant \infty$. Then

$$\left(\int_0^\infty (t^{-\theta} f(t))^q dt/t\right)^{1/q} \leqslant q^{1/q} p^{-1/p} \, (\theta(1-\theta))^{1/q - 1/p} \left(\int_0^\infty (t^{-\theta} f(t))^p \, dt/t\right)^{1/p},$$

where there is equality for $f(t) = \min(1, t)$.

The new feature is that the best constant is determined. This is an unpublished result by Bergh. The inequality goes back to Frank-Pick [1]. (Cf. Borell [1].)

3.14.5—6. We have, in fact, proved more than Theorem 3.5.3 states. We have proved that *if X_i is of class $C_K(\theta_i; \bar{A})$, $i = 0, 1$, then*

$$\bar{X}_{\eta, q} \subset \bar{A}_{\theta, q}.$$

Conversely, if X_i is complete and of class $C_J(\theta_i; \bar{A})$ then

$$\bar{X}_{\eta, q} \supset \bar{A}_{\theta, q}.$$

The assumption that $\bar{A}_{\theta_i,q_i}$ are complete in the last statement of Theorem 3.5.3 is not indispensable. This is a consequence of Holmstedt's [1] formula in 3.6 (see Exercise 15, where a sharper version of the reiteration theorem is found). Holmstedt [1] proved his formula in the quasi-normed case and with $0<q_i\leqslant\infty$.

3.14.7. Theorem 3.7.1 was essentially presented by Lions [3] and Lions-Peetre [1].
The dual of $\bar{A}_{\theta,q}$ when $0<q<1$, has been investigated by Peetre [26]. He showed that $\bar{A}'_{\theta,q}=\bar{A}'_{\theta,1}$ $(0<q<1)$, $\bar{A}$ being a compatible Banach couple with $\Delta(\bar{A})$ dense in A_0 and in A_1.

3.14.8. Compactness theorems of the type:

$$T\colon A_0\to B_0 \quad \textit{(compactly)},$$
$$T\colon A_1\to B_1$$

imply that

$$T\colon \bar{A}_{\theta,q}\to\bar{B}_{\theta,q} \quad \textit{(compactly)},$$

i.e. more general than those in 3.8, have been given by Krasnoselskij [1], Krein-Petunin [1] and Persson [1]. In those theorems, the couple $\bar{B}$ is subject to an approximation condition.

3.14.9. As we noted earlier, Theorem 3.9.1 is related to Krein-Petunin's [1] minimal and maximal scale. The theorem is due to Lions-Peetre [1].

3.14.10—11. These sections are taken over from Peetre-Sparr [1]. Applications of the interpolation results can be found in Chapter 5 and Chapter 7. Related results have been found by Sagher [1] (cf. Exercise 16).

3.14.12. The space $S(\bar{A},\bar{p},\theta)$ is the "espace de moyenne" introduced by Lions-Peetre [1], but with slightly different notation. In fact, let ξ_0 and ξ_1 be any two real numbers such that $\xi_0\xi_1<0$ and $(1-\theta)\xi_0+\theta\xi_1=0$. Making the transformation $t=\tau^{\xi_1-\xi_0}$, we see that the norm on $S(\bar{A},\bar{p},\theta)$ is equivalent to the infimum of

$$\max(\|\tau^{\xi_0}v(\tau)\|_{L^*_{p_0}(A_0)},\ \|\tau^{\xi_1}v(\tau)\|_{L^*_{p_1}(A_1)}),$$

where $a=\int_0^\infty v(\tau)\,d\tau/\tau$. After the additional transformation $\tau=e^x$, we see that the norm on $S(\bar{A},\bar{p},\theta)$ is equivalent to the infimum of

$$\max((\textstyle\int_{-\infty}^{\infty}(e^{\xi_0 x}\|w(x)\|_{A_0})^{p_0}dx)^{1/p_0},\ (\int_{-\infty}^{\infty}(e^{\xi_1 x}\|w(x)\|_{A_1})^{p_1}dx)^{1/p_1}),$$

where $a=\int_{-\infty}^{\infty}w(x)dx$. But this is just the norm on the "espace de moyenne" $S(p_0,\xi_0,A_0;p_1,\xi_1,A_1)$ introduced by Lions-Peetre [1].
By a similar transformation, it will be seen that our space $\underline{S}(\bar{A},\bar{p},\theta)$ is the space $\underline{S}(p_0,\xi_0,A_0;p_1,\xi_1,A_1)$ defined by Lions-Peetre [1].

Theorem 3.12.1 was first given by Peetre with a different proof. He also proved the theorem in case one or both of the numbers p_0 and p_1 is ∞ (see Holmstedt [1]). Thus Theorem 3.12.2 and Corollary 3.12.3 also hold in the case $p_0=\infty$ or $p_1=\infty$.

Writing $\alpha_0=\eta_0-1/p_0$, $\alpha_1=\eta_1-1/p_1$, we see that the norm in our space $\tilde{V}^m(\bar{A},\bar{p},\bar{\eta})$ is equivalent to

$$\max((\int_0^\infty (t^{\alpha_0}\|u(t)\|_{A_0})^{p_0}dt)^{1/p_0}, (\int_0^\infty (t^{\alpha_1}\|u^{(m)}(t)\|_{A_1})^{p_1}dt)^{1/p_1}),$$

which is the norm in the space $V_m(p_0,\alpha_0,A_0;p_1,\alpha_1,A_1)$ introduced in Lions-Peetre [1]. As a consequence our space $T^m(\bar{A},\bar{p},\theta)$ is equal to their space $T_0^m(p_0,\alpha_0,A_0;p_1,\alpha_1,A_1)$ provided that θ is given as in Corollary 3.12.3.

Lions-Peetre [1] are also working with more general trace spaces, using the trace of the derivatives of u. (See Exercise 14.)

Chapter 4

The Complex Interpolation Method

The second of the two interpolation methods which we discuss in detail, the complex method, is treated in this chapter. Our presentation follows the essential points in Calderón [2]. The results are analogous to those obtained for the real method in Chapter 3, but they are frequently more precise here. We make a comparison with the real method in Section 4.7. The proofs in the first sections are more detailed than in the later sections.

Throughout the chapter we consider the category $\mathscr{B}_1$, consisting of compatible Banach couples.

4.1. Definition of the Complex Method

We shall work with analytic functions with values in Banach spaces. The theory of such functions is, as far as we shall need it, parallel to the theory of complex-valued analytic functions.

In this section we introduce two interpolation functors C_θ and C^θ using the theory of vector-valued analytic functions. This will lead to an abstract form of the Riesz-Thorin theorem.

Given a couple $\bar{A}$, we shall consider the space $\mathscr{F}(\bar{A})$ of all functions f with values in $\Sigma(\bar{A})$, which are bounded and continuous on the strip

$$S=\{z: 0\leqslant \operatorname{Re} z\leqslant 1\},$$

and analytic on the open strip

$$S_0=\{z: 0<\operatorname{Re} z<1\},$$

and moreover, the functions $t\to f(j+it)$ $(j=0,1)$ are continuous functions from the real line into A_j, which tend to zero as $|t|\to\infty$. Clearly, $\mathscr{F}(\bar{A})$ is a vector space. We provide $\mathscr{F}$ with the norm

$$\|f\|_{\mathscr{F}}=\max(\sup\|f(it)\|_{A_0}, \sup\|f(1+it)\|_{A_1}).$$

4.1.1. Lemma. *The space $\mathscr{F}$ is a Banach space.*

Proof: Suppose that $\sum_n \|f_n\|_{\mathscr{F}} < \infty$. Since $f_n(z)$ is bounded in $\Sigma(\bar{A})$, we have

$$\|f_n(z)\|_{\Sigma(\bar{A})} \leqslant \max(\sup \|f_n(it)\|_{\Sigma(\bar{A})}, \sup \|f_n(1+it)\|_{\Sigma(\bar{A})}).$$

Since $A_j \subset \Sigma(\bar{A})$, we conclude that

$$\|f_n(z)\|_{\Sigma(\bar{A})} \leqslant \|f_n\|_{\mathscr{F}}.$$

By Lemma 2.3.1, we know that $\Sigma(\bar{A})$ is a Banach space. It follows that $\sum_n f_n$ converges uniformly on S to a function f in $\Sigma(\bar{A})$. Thus f is bounded and continuous on S and analytic in S_0. Furthermore, $\|f_n(j+it)\|_{A_j} \leqslant \|f_n\|_{\mathscr{F}}$ and thus $\sum_n f_n(j+it)$ converges uniformly in t to a limit in A_j, which must coincide with the limit in $\Sigma(\bar{A})$. Therefore, $f(j+it) \in A_j$ and $\sum_n f_n(j+it)$ converges uniformly to $f(j+it)$ in A_j. But then it follows that $f = \mathscr{F}$, and that $\sum_n f_n$ converges to f in $\mathscr{F}$. □

We shall now define the interpolation functor C_θ. *The space $A_{[\theta]} = C_\theta(\bar{A})$ consists of all $a \in \Sigma(\bar{A})$ such that $a = f(\theta)$ for some $f \in \mathscr{F}(\bar{A})$. The norm on $\bar{A}_{[\theta]}$ is*

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathscr{F}} : f(\theta) = a, f \in \mathscr{F}\}.$$

4.1.2. Theorem. *The space $A_{[\theta]}$ is a Banach space and an intermediate space with respect to $\bar{A}$. The functor C_θ is an exact interpolation functor of exponent θ.*

Proof: The linear mapping $f \to f(\theta)$ is a continuous mapping from $\mathscr{F}(\bar{A})$ to $\Sigma(\bar{A})$ since $\|f(\theta)\|_{\Sigma(\bar{A})} \leqslant \|f\|_{\mathscr{F}}$. The kernel of this mapping is $\mathscr{N}_\theta = \{f : f \in \mathscr{F}, f(\theta) = 0\}$. Clearly, $A_{[\theta]}$ is isomorphic and isometric to the quotient space $\mathscr{F}(\bar{A})/\mathscr{N}_\theta$. Since $\mathscr{N}_\theta$ is closed, it follows that $A_{[\theta]}$ is a Banach space. Moreover, since $\|a\|_{\Sigma(\bar{A})} = \|f(\theta)\|_{\Sigma(\bar{A})} \leqslant \|f\|_{\mathscr{F}}$ we obtain $\bar{A}_{[\theta]} \subset \Sigma(\bar{A})$.

Taking $f(z) = \exp(\delta(z-\theta)^2)a$, we also see that $\Delta(\bar{A}) \subset \bar{A}_{[\theta]}$. Thus $\bar{A}_{[\theta]}$ is an intermediate space with respect to $\bar{A}$.

Next, we prove that C_θ is an exact interpolation method of exponent θ. Thus assume that T maps A_j to B_j with norm M_j $(j=0,1)$. Given $a \in A_{[\theta]}$ and $\varepsilon > 0$, there is a function $f \in \mathscr{F}(\bar{A})$, such that $f(\theta) = a$ and $\|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$. Put $g(z) = M_0^{z-1} M_1^{-z} T(f(z))$. g belongs to the class $\mathscr{F}(\bar{B})$. Moreover, $\|g\|_{\mathscr{F}} \leqslant \|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$. But now $g(\theta) = M_0^{\theta-1} M_1^{-\theta} T(a)$ and hence we conclude that $\|T(a)\|_{[\theta]} \leqslant M_0^{1-\theta} M_1^{\theta} \|g\|_{\mathscr{F}} \leqslant M_0^{1-\theta} M_1^{\theta} \|a\|_{[\theta]} + \varepsilon'$, where $\varepsilon' = M_0^{1-\theta} M_1^{\theta} \varepsilon$. This gives the result. □

Now we shall introduce a second complex interpolation method. This is based on a space $\mathscr{G}(\bar{A})$ of analytic functions, defined as follows. The functions g in $\mathscr{G}(\bar{A})$ are defined on the strip S with values in $\Sigma(\bar{A})$. Moreover they have the following properties:

(i) $\|g(z)\|_{\Sigma(\bar{A})} \leqslant c(1+|z|)$,

(ii) g is continuous on S and analytic on S_0,

(iii) $g(j+it_1)-g(j+it_2)$ has values in A_j for all real values of t_1 and t_2 and for $j=0,1$, and

$$\|g\|_{\mathscr{G}}=\max\left(\sup_{t_1,t_2}\left\|\frac{g(it_1)-g(it_2)}{t_1-t_2}\right\|_{A_0},\sup_{t_1,t_2}\left\|\frac{g(1+it_1)-g(1+it_2)}{t_1-t_2}\right\|_{A_1}\right)$$

is finite.

4.1.3. Lemma. *The space $\mathscr{G}(\bar{A})$, reduced modulo constant functions and provided with the norm $\|g\|_{\mathscr{G}}$, is a Banach space.*

Proof: From the conditions it follows easily that if $h\neq 0$ is a real number then

$$\left\|\frac{g(z+ih)-g(z)}{ih}\right\|_{\Sigma(\bar{A})}\leqslant\|g\|_{\mathscr{G}}.$$

Thus we obtain

$$\|g'(z)\|_{\Sigma(\bar{A})}\leqslant\|g\|_{\mathscr{G}}\quad(z\in S).$$

We therefore see that if $\|g\|_{\mathscr{G}}=0$ then g is constant. This implies that $\mathscr{G}$ modulo constants is a normed space. We also see that ($z\in S_0$)

$$\|g(z)-g(0)\|_{\Sigma(\bar{A})}\leqslant|z|\,\|g\|_{\mathscr{G}}.$$

Now suppose that $\sum_n\|g_n\|_{\mathscr{G}}<\infty$. Then $\sum_n(g_n(z)-g_n(0))$ converges uniformly on every compact subset of S_0. The limit $g(z)$ satisfies (i) and (ii). Moreover it follows that the series $\sum_n(g_n(j+it_1)-g_n(j+it_2))$ converges in A_j. Thus $g(j+it_1)-g(j+it_2)\in A_j$ and is the sum of the series $\sum_n(g_n(j+it_1)-g_n(j+it_2))$ in A_j. Therefore $g\in\mathscr{G}$, i.e. $\mathscr{G}$ is complete. □

We now define the space $C^\theta(\bar{A})=\bar{A}^{[\theta]}$ in the following way. *For a given θ such that $0<\theta<1$ we let $\bar{A}^{[\theta]}$ consist of all $a\in\Sigma(\bar{A})$ such that $a=g'(\theta)$ for some $g\in\mathscr{G}(\bar{A})$. The norm on $\bar{A}^{[\theta]}$ is*

$$\|a\|^{[\theta]}=\inf\{\|g\|_{\mathscr{G}}:g'(\theta)=a,g\in\mathscr{G}\}.$$

4.1.4. Theorem. *The space $\bar{A}^{[\theta]}$ is a Banach space and an intermediate space with respect to $\bar{A}$. The functor C^θ is an exact interpolation functor of exponent θ.*

Proof: Since $\|g'(\theta)\|_{\Sigma(\bar{A})}\leqslant\|g\|_{\mathscr{G}}$, we see that the mapping $g\to g'(\theta)$ from $\mathscr{G}$ into $\Sigma(\bar{A})$ is continuous. The kernel $\mathscr{N}^\theta$ of this mapping is closed and the range is $\bar{A}^{[\theta]}$. The norm on $\bar{A}^{[\theta]}$. The norm on $\bar{A}^{[\theta]}$ is the quotient norm on $\mathscr{G}/\mathscr{N}^\theta$. Thus $\bar{A}^{[\theta]}$ is a Banach space. Obviously, $\bar{A}^{[\theta]}\subset\Sigma(\bar{A})$. If $a\in\Delta(\bar{A})$ we take $g(z)=za$ and then we see that $\Delta(\bar{A})\subset\bar{A}^{[\theta]}$.

In order to prove that C^θ is an exact interpolation functor of exponent θ, we assume that $T: A_j \to B_j$ with norm M_j for $j=0,1$. Then we choose a function $g \in \mathscr{G}(\bar{A})$, such that $g'(\theta)=a$, $\|g\|_{\mathscr{G}} \leqslant \|a\|^{[\theta]}+\varepsilon$. Consider the function

$$h(z)=[M_0^{\eta-1}M_1^{-\eta}T(g(\eta))]_{\eta=0}^{\eta=z}-\int_0^z(\log M_0/M_1)M_0^{\eta-1}M_1^{-\eta}T(g(\eta))\,d\eta.$$

The integral is taken along any path in S which connects 0 and z. If the path has all its points in S_0 except 0 and possibly z we may integrate by parts. In fact, if $\eta \in S_0$ we have $d(T(g(\eta)))/d\eta = T(g'(\eta))$ and $g'(\eta)$ is bounded and continuous on S_0. Thus $d(T(g(\eta)))/d\eta$ is continuous on S_0 and has bounded norm in $\Sigma(\bar{B})$. Thus we may integrate by parts, and we obtain, for any path in S,

$$h(z)=\int_0^z M_0^{\eta-1}M_1^{-\eta}T(dg(\eta)),$$

where in general the integral is to be interpreted as a vector-valued Stieltjes integral. It follows that

$$\|h(z)\|_{\Sigma(\bar{B})} \leqslant c\,|z|.$$

Next we note that $T(g(j+it))$ has its values in B_j and is a Lipschitz function in B_j. Thus it follows that

$$\|h(j+it_1)-h(j+it_2)\|_{B_j} \leqslant M_j^{-1}\int_{t_1}^{t_2}\|T(dg(j+it))\|_{B_j}$$

if $t_1<t_2$. But the right hand side is bounded by

$$\int_{t_1}^{t_2}\|dg(j+it)\|_{A_j} \leqslant (t_2-t_1)\,\|g\|_{\mathscr{G}}.$$

It follows that

$$\|h\|_{\mathscr{G}(\bar{B})} \leqslant \|a\|^{[\theta]}+\varepsilon.$$

Now

$$h'(\theta)=M_0^{\theta-1}M_1^{-\theta}\left(\frac{d}{d\eta}T(f(\eta))\right)_{\eta=\theta}=M_0^{\theta-1}M_1^{-\theta}T(a).$$

This proves that $T(a)=M_0^{1-\theta}M_1^{\theta}h'(\theta)\in\bar{B}^{[\theta]}$, and that

$$\|T(a)\|^{[\theta]} \leqslant M_0^{1-\theta}M_1^{\theta}\|a\|^{[\theta]}+\varepsilon'.$$

This gives the result. $\square$

In general, the two spaces $\bar{A}_{[\theta]}$ and $\bar{A}^{[\theta]}$ are not equal. The question of the relation between these two spaces will be discussed in Section 3. The main interest will be attached to the space $\bar{A}_{[\theta]}$. We shall consider the space $\bar{A}^{[\theta]}$ more or less as a technical tool.

4.2. Simple Properties of $\overline{A}_{[\theta]}$

We shall prove two simple results concerning inclusion and density properties of the spaces $(A_0, A_1)_{[\theta]}$.

4.2.1. Theorem. *We have*

(a) $(A_0, A_1)_{[\theta]} = (A_1, A_0)_{[1-\theta]}$ *(with equal norms),*

(b) $A_1 \subset A_0 \Rightarrow (A_0, A_1)_{[\theta_1]} \subset (A_0, A_1)_{[\theta_0]}$ *if* $\theta_0 < \theta_1$,

(c) $(A, A)_{[\theta]} = A$ *if* $0 < \theta < 1$.

Proof: In order to prove (a), we have only to note that $f(z) \in \mathscr{F}(A_0, A_1)$ if and only if $f(1-z) \in \mathscr{F}(A_1, A_0)$. Using (a), we shall obtain (b) if we can prove that $A_0 \subset A_1$ implies $(A_0, A_1)_{[\theta]} \subset (A_0, A_1)_{[\tilde{\theta}]}$ when $\theta < \tilde{\theta}$. If $a \in (A_0, A_1)_{[\theta]}$ we can choose $f \in \mathscr{F}(\bar{A})$ so that $f(\theta) = a$, $\|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$. Put $\theta = \lambda\tilde{\theta}$ where $0 \leqslant \lambda < 1$ and $\varphi(z) = f(\tilde{\theta}z)\exp(\varepsilon(z^2 - \lambda^2))$. Writing $B_1 = (A_0, A_1)_{[\tilde{\theta}]}$, we have $\|f(\tilde{\theta} + it)\|_{B_1} \leqslant \|f\|_{\mathscr{F}(\bar{A})}$. It follows that

$$\|\varphi\|_{\mathscr{F}(A_0, B_1)} \leqslant (\|a\|_{[\theta]} + \varepsilon)\exp\varepsilon.$$

But now $\varphi(\lambda) = a$ and $(A_0, B_1)_{[\lambda]} \subset (B_1, B_1)_{[\lambda]} = B_1$ (immediate), and thus

$$\|a\|_{[\tilde{\theta}]} \leqslant c\,\|\varphi(\lambda)\|_{(A_0, B_1)_{[\lambda]}} \leqslant c\,\|\varphi\|_{\mathscr{F}(A_0, B_1)}.$$

It follows that $\|a\|_{[\tilde{\theta}]} \leqslant c\,\|a\|_{[\theta]}$. (c) is obvious. □

4.2.2. Theorem. *Let* $0 \leqslant \theta \leqslant 1$. *Then*

(a) $\Delta(\bar{A})$ *is dense in* $\bar{A}_{[\theta]}$;

(b) *if* A_j^0 *denotes the closure of* $\Delta(\bar{A})$ *in* A_j *we have*

$$(A_0, A_1)_{[\theta]} = (A_0^0, A_1)_{[\theta]} = (A_0, A_1^0)_{[\theta]} = (A_0^0, A_1^0)_{[\theta]};$$

(c) *the space* $B_j = \bar{A}_{[j]}$ $(j = 0, 1)$ *is a closed subspace of* A_j *and the norms coincide in* B_j;

(d) $(A_0, A_1)_{[\theta]} = (B_0, B_1)_{[\theta]}$, *with* B_j *as in* (c).

The proof of Theorem 4.2.2 is based on the following lemma.

4.2.3. Lemma. *Let* $\mathscr{F}_0(\bar{A})$ *be the space of all linear combinations of functions of the form*

$$\exp(\delta z^2)\textstyle\sum_{n=1}^{N} a_n \exp(\lambda_n z)$$

where $a_n \in \Delta(\bar{A})$, λ_n *real and* $\delta > 0$. *Then* $\mathscr{F}_0(\bar{A})$ *is dense in* $\mathscr{F}(\bar{A})$.

Proof: Since $\|\exp(\delta z^2)f(z)-f(z)\|_{\mathscr{F}}\to 0$ as $\delta\to 0$ ($\delta>0$) for all $f\in\mathscr{F}(\bar{A})$, it is sufficient to show that all functions $g(z)=\exp(\delta z^2)f(z)$ with $f\in\mathscr{F}(\bar{A})$ can be approximated by functions in $\mathscr{F}_0(\bar{A})$. Put

$$g_n(z)=\sum_k g(z+2\pi ikn), \quad (n\geq 1).$$

Clearly g_n is analytic on S_0, continuous on S with values in $\Sigma(\bar{A})$. Moreover, g_n is periodic with period $2\pi in$, and $g_n(j+it)\in A_j$ for $j=0,1$. Furthermore $\|g_n(j+it)-g(j+it)\|_{A_j}\to 0$ as $n\to\infty$ uniformly on every compact set of t-values and $\|g_n(j+it)\|_{A_j}$ is bounded as a function of n and t. It follows that, for all $s>0$, we have $\exp(sz^2)g_n(z)$ in the space $\mathscr{F}(\bar{A})$. Therefore, we can find s and n so that

$$\|\exp(sz^2)g_n(z)-g(z)\|_{\mathscr{F}}<\varepsilon.$$

But now $g_n(z)$ can be represented by a Fourier series

$$(1) \qquad g_n(z)=\sum_k a_{kn}e^{kz/n}, \quad z=s+it,$$

where

$$a_{kn}=(2\pi nm)^{-1}\int_{-\pi nm}^{\pi nm} g_n(s+it)e^{-k(s+it)/n}\,dt.$$

Note that, by periodicity, the integral is independent of m. It is also independent of s. In fact, the integrand is analytic and bounded in $\Sigma(\bar{A})$. Thus the values of the integral for two values of s will differ very little if m is chosen large, due to the presence of the factor $1/m$. But the integral is independent of m. Thus the integral has the same value for the two given values of s. It follows that

$$a_{kn}=(2\pi n)^{-1}\int_{-n\pi}^{n\pi} g_n(j+it)e^{-(j+it)k/n}\,dt, \quad j=0,1.$$

Then we have $a_{kn}\in\Delta(\bar{A})$. Now we consider the $(C,1)$-means of the sum (1), i.e. we consider

$$\sigma_m g_n(z)=\sum_{|k|\leq m}\left(1-\frac{|k|}{m+1}\right)a_{kn}e^{kz/n}.$$

Then $\|\sigma_m g_n(j+it)-g_n(j+it)\|_{A_j}\to 0$ as $m\to\infty$, uniformly in n. Thus $\|\exp(sz^2)(\sigma_m g_n-g_n)\|_{\mathscr{F}}\to 0$ as $m\to\infty$ and so

$$\|\exp(sz^2)\sigma_m g_n-g\|_{\mathscr{F}}<2\varepsilon.$$

But $\exp(sz^2)\sigma_m g_n\in\mathscr{F}_0(\bar{A})$. This proves the lemma. □

Proof of Theorem 4.2.2: (a) If $a\in\bar{A}_{[\theta]}$ there exists a function $f\in\mathscr{F}(\bar{A})$, such that $f(\theta)=a$. Then there exists $g\in\mathscr{F}_0(\bar{A})$, such that $\|f-g\|_{\mathscr{F}}<\varepsilon$. Therefore $\|a-g(\theta)\|_{[\theta]}<\varepsilon$ and since $g(\theta)\in\Delta(\bar{A})$ the conclusion follows.

(b) Follows at once from (a).

(c) Clearly, $B_0 \subset A_0$ and $B_1 \subset A_1$. Let us prove that the norm on B_0 coincides with the norm on A_0. Take $a \in B_0$. Then we can find $a_1 \in \Delta(\bar{A})$, such that $\|a-a_1\|_{B_0} < \varepsilon$. Consider $f_n(z) = a_1 e^{z^2-nz} \in \mathscr{F}(\bar{A})$. Then $f_n(0) = a_1$ and $\|f_n\|_{\mathscr{F}} \leqslant \|a_1\|_{A_0} + e^{1-n}\|a_1\|_{A_1}$. Since $\|a_1\|_{B_0} \leqslant \|f_n\|_{\mathscr{F}}$ for all n, we conclude that $\|a_1\|_{B_0} \leqslant \|a_1\|_{A_0}$. But $\|a\|_{A_0} \leqslant \|a\|_{B_0}$ and so $\|a-a_1\|_{A_0} \leqslant \|a-a_1\|_{B_0} < \varepsilon$. Thus it follows that $\|a\|_{B_0} \leqslant \varepsilon + \|a_1\|_{B_0} \leqslant 2\varepsilon + \|a\|_{A_0}$, and hence $\|a\|_{B_0} \leqslant \|a\|_{A_0}$. This proves that $\|a\|_{B_0} = \|a\|_{A_0}$.

(d) Obviously, (d) follows if we can prove that $\mathscr{F}(\bar{A}) = \mathscr{F}(\bar{B})$. Evidently, $\mathscr{F}(\bar{B}) \subset \mathscr{F}(\bar{A})$. But if $f(z) \in \mathscr{F}(\bar{A})$ then $f(j+it) \in B_j$ (by the definition of B_j). Thus $f(z) \in \mathscr{F}(\bar{B})$, proving (d). ☐

4.3. The Equivalence Theorem

We shall now study the relation between the two complex interpolation methods C_θ and C^θ. We shall prove that they are equivalent when applied to certain couples.

4.3.1. Theorem (The complex equivalence theorem). *For any couple* $\bar{A} = (A_0, A_1)$, *we have*

$$\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]} \quad \textit{and} \quad \|a\|^{[\theta]} \leqslant \|a\|_{[\theta]}.$$

If at least one of the two spaces A_0 *and* A_1 *is reflexive and if* $0 < \theta < 1$, *then*

$$\bar{A}_{[\theta]} = \bar{A}^{[\theta]} \quad \textit{and} \quad \|a\|^{[\theta]} = \|a\|_{[\theta]}.$$

Proof: Take $a \in \bar{A}_{[\theta]}$, and choose $f \in \mathscr{F}(\bar{A})$ so that $f(\theta) = a$ and $\|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$. Then put $g(z) = \int_0^z f(\zeta)\,d\zeta$. Then it is readily seen that $g \in \mathscr{G}(\bar{A})$, and that $\|g\|_{\mathscr{G}} \leqslant \|f\|_{\mathscr{F}}$. Moreover, $g'(\theta) = f(\theta) = a$. Consequently, $\|a\|^{[\theta]} \leqslant \|g\|_{\mathscr{G}} \leqslant \|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$, proving the first part of the theorem.

The proof of the second part is much deeper. Let us denote by $P_j, j=0,1$, the Poisson kernels for the strip S. They can be obtained from the Poisson kernel for the half-plane by means of a conformal mapping. Explicitly, we have that

$$P_j(s+it,\tau) = \frac{e^{-\pi(\tau-t)}\sin\pi s}{\sin^2\pi s + (\cos\pi s - e^{ij\pi-\pi(\tau-t)})^2}, \qquad j=0,1.$$

4.3.2. Lemma. *If* $f \in \mathscr{F}(\bar{A})$ *we have*

(i) $\log\|f(\theta)\|_{[\theta]} \leqslant \sum_{j=0,1} \int_{-\infty}^{\infty} \log\|f(j+i\tau)\|_{A_j} P_j(\theta,\tau)\,d\tau$

(ii) $\|f(\theta)\|_{[\theta]} \leqslant \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \|f(i\tau)\|_{A_0} P_0(\theta,\tau)\,d\tau\right)^{1-\theta} \cdot \left(\frac{1}{\theta}\|f(1+i\tau)\|_{A_1} P_1(\theta,\tau)\,d\tau\right)^{\theta}$

(iii) $\|f(\theta)\|_{[\theta]} \leqslant \sum_{j=0,1} \int_{-\infty}^{\infty} \|f(j+i\tau)\|_{A_j} P_j(\theta,\tau)\,d\tau.$

4.3.3. Lemma. *If* $f \in \mathscr{G}(\bar{A})$ *and if* f *has the property that*

$$\frac{1}{h}(f(it+ih)-f(it))$$

converges in A_0 *on a set of positive measure as* $h \to 0$ *(h real), then* $f'(\theta) \in \bar{A}_{[\theta]}$ *for* $0<\theta<1$.

Proof of Lemma 4.3.2: Let φ_j be an infinitely differentiable bounded function such that

$$\varphi_j(t) \geqslant \log \| f(j+it)\|_{A_j}, \quad (j=0,1).$$

Let $\Phi(z)$ be an analytic function such that

$$\operatorname{Re}\Phi(z) = \int_{-\infty}^{\infty} \varphi_0(\tau) P_0(z,\tau)\,d\tau + \int_{-\infty}^{\infty} \varphi_1(\tau) P_1(z,\tau)\,d\tau.$$

Then $\operatorname{Re}\Phi(j+it)=\varphi_j(it)$, $j=0,1$ and Φ is continuous and bounded on S. Thus $\exp(-\Phi)\cdot f \in \mathscr{F}(\bar{A})$. Since

$$\|\exp(-\Phi(j+it))\cdot f(j+it)\|_{A_j} \leqslant \exp(-\varphi_j(t))\cdot \|f(j+it)\|_{A_j} \leqslant 1$$

it follows that $\|\exp(-\Phi)f\|_{\mathscr{F}} \leqslant 1$. Thus

$$\|\exp(-\Phi)f\|_{[\theta]} \leqslant 1.$$

Therefore we conclude that

$$\log \|f(\theta)\|_{[\theta]} \leqslant \operatorname{Re}\Phi(\theta) = \sum_{j=0,1} \int_{-\infty}^{\infty} \varphi_j(i\tau) P_j(\theta,\tau)\,d\tau.$$

Taking decreasing sequences of functions φ_0 and φ_1 converging to $\log\|f(it)\|_{A_0}$ and $\log\|f(1+it)\|_{A_1}$ respectively, we get (i). In order to get (ii), we apply Jensen's inequality with the exponential function to (i). (Note that $\int_{-\infty}^{\infty} P_0(\theta,\tau)\,d\tau = 1-\theta$ and $\int_{-\infty}^{\infty} P_1(\theta,\tau)\,d\tau=\theta$.) Finally, (iii) follows from (ii) by the inequality between the arithmetic and the geometric means. □

Proof of Lemma 4.3.3: Put

$$f_n(z) = (i/n)^{-1}(f(z+i/n)-f(z)).$$

Then $\|f_n(it)-f_m(it)\|_{A_0} \to 0$ as $n,m\to\infty$ for all t on a set E of positive measure. Further, we have that $\exp(\varepsilon z^2) f_n(z) \in \mathscr{F}(\bar{A})$ for all $\varepsilon>0$. From Lemma 4.3.2 we obtain

$$\begin{aligned}&\log \|e^{\varepsilon\theta^2}(f_n(\theta)-f_m(\theta))\|_{[\theta]}\\ &\quad \leqslant \sum_{j=0,1} \int_{-\infty}^{\infty} \log \|e^{-\varepsilon(j+i\tau)^2}(f_n(j+i\tau)-f_m(j+i\tau))\|_{A_j} P_j(\theta,\tau)\,d\tau.\end{aligned}$$

Since $\|f_n(j+it)-f_m(j+it)\|_{A_j} \leqslant 2\|f\|_{\mathscr{G}}$, and since $\|f_n(it)-f_m(it)\|_{A_0} \to 0$ for all $t \in E$, we see that the right hand side tends to $-\infty$ as $n, m \to \infty$. (Note that $P_0 > 0$.) Thus $\log\|\exp(\varepsilon\theta^2)(f_n(\theta)-f_m(\theta))\|_{[\theta]} \to -\infty$ as $n, m \to \infty$, and, consequently, $\|f_n(\theta)-f_m(\theta)\|_{[\theta]} \to 0$. Therefore, $f_n(\theta)$ converges in $\bar{A}_{[\theta]}$. But $f_n(\theta)$ converges in $\Sigma(\bar{A})$ to $f'(\theta)$. Hence $f'(\theta) \in \bar{A}_{[\theta]}$. □

Completion of the proof of Theorem 4.3.1: We shall prove that if one of the spaces A_0 and A_1 is reflexive then $\bar{A}^{[\theta]} \subset \bar{A}_{[\theta]}$ and $\|a\|_{[\theta]} \leqslant \|a\|^{[\theta]}$. By Theorem 4.2.1 we may assume that A_0 is reflexive.

If $f \in \mathscr{G}(\bar{A})$ then $f(it)$ is continuous and therefore its range lies in a separable subspace V of A_0. Put $f_n(z) = (f(z+i/n) - f(z)) \cdot n/i$ and let $R_m(t)$ be the weak closure of the set $\{f_n(it): n \geqslant m\}$. Put $R(t) = \bigcap_m R_m(t)$. Then $R_m(t)$ and $R(t)$ are bounded (uniformly in t and m) subsets of A_0. Since $R_m(t)$ is bounded and weakly closed, and since the unit sphere of A_0 is weakly compact (A_0 is reflexive), we conclude that $R_m(t)$ is weakly compact. Therefore $R(t)$ is non-empty. Now let $g(t)$ be a function such that $g(t) \in R(t)$ for each t. Since $R(t) \subset V$, the range of g is separable.

We shall prove that

$$(1) \qquad f(it) = f(0) + i\int_0^t g(\tau)\,d\tau\,.$$

Let L be a continuous linear functional on A_0, and put $\varphi(t) = -iL(f(it))$. Then the assumption $f \in \mathscr{G}(\bar{A})$ implies that φ is Lipschitz continuous. Moreover,

$$L(f_n(it)) = n(\varphi(t+1/n) - \varphi(t))\,.$$

The image of $R_m(t)$ under L is the closure of the set $\{n(\varphi(t+1/n)-\varphi(t)): n \geqslant m\}$. The image of $R(t)$ is contained in the intersection of these sets. If φ is differentiable at the point t, the image of $R(t)$ under L will therefore be $\{\varphi'(t)\}$. Consequently, we have $L(g(t)) = \varphi'(t)$ whenever $\varphi'(t)$ exists. But φ is Lipschitz continuous. Therefore, $\varphi'(t)$ exists almost everywhere and is measurable. It follows that $L(g(t))$ exists almost everywhere and is measurable. Since the range of g is separable, it follows that g is strongly measurable. Since the sets $R(t)$ are all contained in a bounded set, $g(t)$ is also bounded. Thus

$$L(f(it) = i\varphi(t) = i\varphi(0) + i\int_0^t \varphi'(\tau)\,d\tau = L(f(0)) + i\int_0^t L(g(\tau))\,d\tau.$$

This implies (1).

From (1) we see that $f(it)$ has a strong derivative almost everywhere. Thus Lemma 4.3.3 implies that $f'(\theta) \in \bar{A}_{[\theta]}$. But $f'(\theta)$ is a typical element in $\bar{A}^{[\theta]}$. Thus $\bar{A}^{[\theta]} \subset \bar{A}_{[\theta]}$. More precisely, if $a \in \bar{A}^{[\theta]}$ we can choose $f \in \mathscr{G}(\bar{A})$, such that $f'(\theta) = a$ and $\|f\| \leqslant \|a\|^{[\theta]} + \varepsilon$. Consider the function

$$h_n(z) = \exp(\varepsilon z^2) \cdot f_n(z).$$

Then $h_n \in \mathscr{F}(\bar{A})$ and $\|h_n\|_{\mathscr{F}} \leqslant e^{\varepsilon}\|f\|_{\mathscr{G}}$. Thus $\|h_n(\theta)\|_{[\theta]} \leqslant e^{\varepsilon}(\|a\|^{[\theta]}+\varepsilon)$. But $\|h_n(\theta)-\exp(\varepsilon\theta^2)a\|_{[\theta]}$ tends to zero as $n\to\infty$. Letting $n\to\infty$ and then $\varepsilon\to 0$ in the inequality

$$\|a\|_{[\theta]} \leqslant \exp(-\varepsilon\theta^2)(\|h_n(\theta)-\exp(\varepsilon\theta^2)a\|_{[\theta]}+e^{\varepsilon}(\|a\|^{[\theta]}+\varepsilon)),$$

we clearly obtain

$$\|a\|_{[\theta]} \leqslant \|a\|^{[\theta]}. \quad \square$$

4.4. Multilinear Interpolation

We prove two theorems concerning multilinear interpolation. The second of these will be applied later.

4.4.1. Theorem. *Let* $\bar{A}^{(\nu)}$ $(\nu=1,2,\ldots,n)$ *and* $\bar{B}$ *be compatible Banach couples. Assume that* $T\colon \Sigma^{\oplus}_{1\leqslant\nu\leqslant n}\Delta(\bar{A}^{(\nu)})\to\Delta(\bar{B})$ *is multilinear and*

$$\begin{aligned}\|T(a_1,\ldots,a_n)\|_{B_0} &\leqslant M_0\textstyle\prod_{\nu=1}^{n}\|a_\nu\|_{A_0^{(\nu)}}\\ \|T(a_1,\ldots,a_n)\|_{B_1} &\leqslant M_1\textstyle\prod_{\nu=1}^{n}\|a_\nu\|_{A_1^{(\nu)}}.\end{aligned} \qquad (a_\nu\in\Delta(\bar{A}^{(\nu)}))$$

Then T may be uniquely extended to a multilinear mapping from $\sum^{\oplus}_{1\leqslant\nu\leqslant n}\bar{A}^{(\nu)}_{[\theta]}$ *to* $\bar{B}_{[\theta]}$ *with norm at most* $M_0^{1-\theta}M_1^{\theta}$ $(0\leqslant\theta\leqslant 1)$.

Proof (cf. the proof of Theorem 4.1.2): Put

$$g(z)=M_0^{z-1}M_1^{z}\,T(f_1(z),\ldots,f_n(z)),$$

where $f_\nu\in\mathscr{F}_\nu=\mathscr{F}(\bar{A}^{(\nu)})$. This is at first only defined for functions f_ν with values in $\Delta(\bar{A}^{(\nu)})$, but can be extended to $\sum^{\oplus}_{1\leqslant\nu\leqslant n}\mathscr{F}_\nu$ by Lemma 4.2.3. Now $g\in\mathscr{F}(\bar{B})$ and $\|g\|_{\mathscr{F}}\leqslant\prod_{1\leqslant\nu\leqslant n}\|f_\nu\|_{\mathscr{F}_\nu}$. Take $a=(a_1,\ldots,a_n)=(f_1(\theta),\ldots,f_n(\theta))\in\sum^{\oplus}_{1\leqslant\nu\leqslant n}\Delta(\bar{A}^{(\nu)})$ with $f_\nu\in\mathscr{F}_\nu$ and $\prod_{1\leqslant\nu\leqslant n}\|f_\nu\|_{\mathscr{F}_\nu}\leqslant\prod_{1\leqslant\nu\leqslant n}\|a_\nu\|_{\bar{A}^{(\nu)}_{[\theta]}}+\varepsilon$, $\varepsilon>0$ arbitrary. It follows that

$$\begin{aligned}\|Ta\|_{[\theta]} &\leqslant M_0^{1-\theta}M_1^{\theta}\|g\|_{\mathscr{F}}\leqslant M_0^{1-\theta}M_1^{\theta}\textstyle\prod_{\nu=1}^{n}\|f_\nu\|_{\mathscr{F}_\nu}\\ &\leqslant M_0^{1-\theta}M_1^{\theta}(\textstyle\prod_{\nu=1}^{n}\|a_\nu\|_{\bar{A}^{(\nu)}_{[\theta]}}+\varepsilon),\end{aligned}$$

and, since ε is arbitrary and $\Delta(\bar{A}^{(\nu)})$ is dense in $\bar{A}^{(\nu)}_{[\theta]}$, $1\leqslant\nu\leqslant n$, this yields the desired conclusion. $\square$

4.4.2. Theorem. *Let* $\bar{A}^{(\nu)}$, $\nu=1,\dots,n$ *and* $\bar{B}$ *be compatible Banach couples. Assume that* $T\colon \Sigma(\bar{A}^{(1)})\oplus\Sigma^{\oplus}_{2\leqslant\nu\leqslant n}\Delta(\bar{A}^{(\nu)})\to\Sigma(\bar{B})$ *is multilinear, the restriction to* $\bar{A}_i^{(1)}$ *having values in* B_i, $i=0,1$, *and*

$$\|T(a^{(1)},\dots,a^{(n)})\|_{B_0}\leqslant M_0\prod_{\nu=1}^{n}\|a^{(\nu)}\|_{A_0^{(\nu)}}\qquad(a^{(1)}\in A_0^{(1)}),$$

$$\|T(a^{(1)},\dots,a^{(n)})\|_{B_1}\leqslant M_1\prod_{\nu=1}^{n}\|a^{(\nu)}\|_{A_1^{(\nu)}}\qquad(a^{(1)}\in A_1^{(1)}).$$

Then T may be extended uniquely to a multilinear mapping from

$$(\bar{A}^{(1)})^{[\theta]}\oplus\sum\nolimits^{\oplus}_{2\leqslant\nu\leqslant n}\bar{A}^{(\nu)}_{[\theta]}$$

to $\bar{B}^{[\theta]}$ *with norm at most* $M_0^{1-\theta}M_1^{\theta}$, $0<\theta<1$.

Proof (cf. the proof of Theorem 4.1.4): Put

$$h(z)=\int_0^z M_0^{\eta-1}M_1^{-\eta}\,T(g_1'(\eta),f_2(\eta),\dots,f_n(\eta))\,d\eta,$$

where $g_1\in\mathscr{G}_1=\mathscr{G}(\bar{A}^{(1)})$, $f_\nu\in\mathscr{F}_\nu=\mathscr{F}(\bar{A}^{(\nu)})$ with values in $\Delta(\bar{A}^{(\nu)})$, $2\leqslant\nu\leqslant n$, the integration being along any curve connecting 0 and z and lying in S_0 with the exception of (possibly both) the endpoints. We may extend (multilinearly) the definition of h to $\mathscr{G}_1+\sum^{\oplus}_{2\leqslant\nu\leqslant n}\mathscr{F}_\nu$ by Lemma 4.2.3. Arguing as in the proof of Theorem 4.1.4, we may write

$$h(z)=\int_0^z M_0^{\eta-1}M_1^{-\eta}\,T(dg_1(\eta),f_2(\eta),\dots,f_n(\eta)),$$

for any path in S. Clearly, $\|h(z)\|_{\Sigma(\bar{B})}\leqslant C|z|$, and, for $t_1<t_2$, $j=0,1$, we obtain

$$\begin{aligned}&\|h(j+it_2)-h(j+it_1)\|_{B_j}\\&\quad\leqslant M_j^{-1}\int_{t_1}^{t_2}\|T(dg_1(j+it),f_2(j+it),\dots,f_n(j+it))\|_{A_j}\\&\quad\leqslant(t_2-t_1)\|g_1\|_{\mathscr{G}_1}\prod\nolimits_{2\leqslant\nu\leqslant n}\|f_\nu\|_{\mathscr{F}_\nu},\end{aligned}$$

i.e., $h\in\mathscr{G}=\mathscr{G}(\bar{B})$ and $\|h\|_{\mathscr{G}}\leqslant\|g_1\|_{\mathscr{G}_1}\prod_{2\leqslant\nu\leqslant n}\|f_\nu\|_{\mathscr{F}_\nu}$. Choosing $g_1\in\mathscr{G}_1$ and $f_\nu\in\mathscr{F}_\nu$ such that $a=(a_1,\dots,a_n)=(g_1(\theta),f_2(\theta),\dots,f_n(\theta))\in(\bar{A}^{(1)})^{[\theta]}\oplus\sum^{\oplus}_{2\leqslant\nu\leqslant n}\Delta(\bar{A}^{(\nu)})$ with $\|g_1\|_{\mathscr{G}_1}\prod_{2\leqslant\nu\leqslant n}\|f_\nu\|_{\mathscr{F}_\nu}\leqslant\|a_1\|_{(\bar{A}^{(1)})^{[\theta]}}\prod_{2\leqslant\nu\leqslant n}\|a_\nu\|_{\bar{A}^{(\nu)}_{[\theta]}}+\varepsilon$, $\varepsilon>0$ arbitrary, it follows that $T(a)=M_0^{1-\theta}M_1^{\theta}h'(\theta)\in\bar{B}^{[\theta]}$ and

$$\begin{aligned}\|T(a)\|_{\bar{B}^{[\theta]}}&\leqslant M_0^{1-\theta}M_1^{\theta}\|h\|_{\mathscr{G}}\leqslant M_0^{1-\theta}M_1^{\theta}\|g_1\|_{\mathscr{G}_1}\prod\nolimits_{2\leqslant\nu\leqslant n}\|f_\nu\|_{\mathscr{F}_\nu}\\&\leqslant M_0^{1-\theta}M_1^{\theta}(\|a_1\|_{(\bar{A}^{(1)})^{[\theta]}}\prod\nolimits_{2\leqslant\nu\leqslant n}\|a_\nu\|_{\bar{A}^{(\nu)}_{[\theta]}}+\varepsilon).\end{aligned}$$

Because $\varepsilon>0$ is arbitrary and $\Delta(\bar{A}^{(\nu)})$ is dense in $\bar{A}^{(\nu)}_{[\theta]}$, $2\leqslant\nu\leqslant n$, the desired conclusion follows. □

4.5. The Duality Theorem

We shall now characterize the dual $\bar{A}_{[\theta]}'$ of the interpolation space $\bar{A}_{[\theta]}$. We utilize a multilinear result: Theorem 4.4.2, the equivalence Theorem 4.3.1 and a result concerning the dual $L_1(A)'$ of $L_1(A)$: the space of integrable functions on $\mathbb{R}$ with values in the Banach space A.

4.5.1. Theorem (The duality theorem). *Assume that $\bar{A}=(A_0, A_1)$ is a compatible Banach couple, and that $\Delta(\bar{A})$ is dense in both A_0 and A_1. Then*

$$(A_0, A_1)_{[\theta]}' = (A_0', A_1')^{[\theta]} \quad \textit{(equal norms; } 0<\theta<1).$$

4.5.2. Corollary. *Suppose that, in addition, at least one of the spaces A_0 and A_1 is reflexive. Then*

$$(A_0, A_1)_{[\theta]}' = (A_0', A_1')_{[\theta]} \quad \textit{(equal norms; } 0<\theta<1).$$

The corollary follows at once by the equivalence and duality theorems.
For the proof of the duality theorem, we need a lemma.

4.5.3. Lemma. *Let A be a Banach space. The dual of $L_1(A)$: the space of integrable A-valued functions on $\mathbb{R}$, is the space $\Lambda(A')$: the space of all functions g of bounded variation on $\mathbb{R}$ such that $g(s)-g(t)\in A'$ for all s and t, and for which*

$$\|g\|_{\Lambda(A')} = \sup_{s\neq t} \|(s-t)^{-1}(g(s)-g(t))\|_{A'}$$

is finite. The duality is given by

$$(1) \qquad \langle g, f\rangle = \int \langle dg(x), f(x)\rangle,$$

or, if $f(x)=h(x)\cdot a$ and h is scalar-valued, by

$$(2) \qquad \langle g, f\rangle = \int h(x)\frac{d}{dx}\langle g(x), a\rangle\, dx.$$

Proof of Thorem 4.5.1: First, consider the bilinear functional $\langle a', a\rangle$ defined on $\Sigma(\bar{A}')\oplus\Delta(\bar{A})$ (cf. Theorem 2.7.1), and use the density assumption. From Theorem 4.4.2 we infer that it has a unique extension to $\bar{A}'^{[\theta]}\oplus\bar{A}_{[\theta]}$ such that, for $a'\in\bar{A}'^{[\theta]}$ and $a\in\bar{A}_{[\theta]}$,

$$|\langle a', a\rangle| \leq \|a'\|_{\bar{A}'^{[\theta]}}\|a\|_{\bar{A}_{[\theta]}}.$$

Thus, if $a'\in\bar{A}'^{[\theta]}$ then $a'\in\bar{A}_{[\theta]}'$ and $\|a'\|_{\bar{A}_{[\theta]}'}\leq\|a'\|_{\bar{A}'^{[\theta]}}$.
Secondly, let $l\in\bar{A}_{[\theta]}'$, i.e.,

$$|l(a)|\leq\|l\|_{\bar{A}_{[\theta]}'}\|a\|_{\bar{A}_{[\theta]}} \qquad (a\in\bar{A}_{[\theta]}).$$

Since $\bar{A}_{[\theta]}$ is identified with a quotient space of $\mathscr{F}(\bar{A})$, l may be defined on the whole of $\mathscr{F}(\bar{A})$ with the same norm. Then the mapping

$$\lambda: (f_0, f_1) \mapsto l(f) = l(a) \qquad (f(\theta) = a)$$

defined on $E = \{(f_0, f_1) \in L_1(A_0) \oplus L_1(A_1) \mid \exists f \in \mathscr{F}(\bar{A}); f_j(\tau) = f(j + i\tau) P_j(0, \tau)\}$ (P_j being the Poisson kernels in Section 4.3) is continuous in the norm $\|f_0\|_{L_1(A_0)} + \|f_1\|_{L_1(A_1)}$, for we have $(f(\theta) = a)$

$$|\lambda(f_0, f_1)| = |l(a)| \leq \|l\|_{\bar{A}_{[\theta]}'} \|a\|_{\bar{A}_{[\theta]}} \leq \|l\|_{\bar{A}_{[\theta]}'} (\|f_0\|_{L_1(A_0)} + \|f_1\|_{L_1(A_1)})$$

by Lemma 4.3.2. E is a linear subspace of $L_1(A_0) \oplus L_1(A_1)$. Thus, by the Hahn-Banach theorem and Lemma 4.5.3, there is $(g_0, g_1) \in \Lambda(A_0') \oplus \Lambda(A_1')$ such that

$$\max(\|g_0\|_{\Lambda(A_0')}, \|g_1\|_{\Lambda(A_1')}) \leq \|l\|_{\bar{A}_{[\theta]}'}$$

and

$$\lambda(f_0, f_1) = \langle g_0, f_0 \rangle + \langle g_1, f_1 \rangle, \qquad (f_0, f_1) \in L_1(A_0) \oplus L_1(A_1).$$

Thus, taking $f_j(\tau) = f(j + i\tau) P_j(\theta, \tau)$, we obtain

$$l(a) = \langle g_0, f(i\tau) P_0(\theta, \tau) \rangle + \langle g_1, f(1 + i\tau) P_1(\theta, \tau) \rangle$$

for $f \in \mathscr{F}(\bar{A})$ with $f(\theta) = a$. It remains to prove that $g_j(\tau) = g(j + i\tau)$ are the boundary values of a function $g \in \mathscr{G}(\bar{A}')$ such that $l(a) = \langle g'(\theta), a \rangle$ for $a \in \bar{A}_{[\theta]}$. In order to find g, take $a \in \Delta(\bar{A})$, and let $f(z) = h(z) \cdot a \in \mathscr{F}(\bar{A})$, h being complex-valued. Obviously, by the representation formula (2),

$$\begin{aligned} l(f) = h(\theta) l(a) &= \langle g_0, h(i\tau) P_0(\theta, \tau) a \rangle + \langle g_1, h(1 + i\tau) P_1(\theta, \tau) a \rangle \\ &= \int h(i\tau) P_0(\theta, \tau) \frac{d}{d\tau} \langle g_0(\tau), a \rangle \, d\tau + \int h(1 + i\tau) P_1(\theta, \tau) \frac{d}{d\tau} \langle g_1(\tau), a \rangle \, d\tau. \end{aligned}$$

Note that $h(\theta) = 0$ implies that the sum of the integrals vanishes. We shall see that this fact implies the existence of a function $g \in \mathscr{G}(\bar{A}')$ with the desired properties.

First we map the strip $0 < \operatorname{Re} z < 1$ conformally onto the unit disc $|w| < 1$, so that the origin is the image of the point θ, using, for instance, the mapping

$$\mu(z) = \frac{\exp(i\pi z) - \exp(i\pi\theta)}{\exp(i\pi z) - \exp(-i\pi\theta)}.$$

Let $\tilde{k}_a$ be the function defined on $|w| = 1$ except at the two points 1 and $\exp(2\pi i\theta)$ by the formula

$$(\tilde{k}_a \circ \mu)(j + i\tau) = \frac{d}{d\tau} \langle g_j(\tau), a \rangle, \qquad j = 0, 1.$$

Then

(3) $$\int_{|w|=1}\tilde{h}(w)\tilde{k}_a(w)\,dw=0$$

if $\tilde{h}$ is given by the formula $\tilde{h}\circ\mu=h$, where $h(\theta)=0$. We can for instance take $h(z)=(\mu(z))^n\exp(\varepsilon z^2)$, with $\varepsilon>0$ and $n=1,2,\ldots$. Letting $\varepsilon\to 0$ we see that (3) holds for $\tilde{h}(w)=w^n, n=1,2,\ldots$. Thus the Fourier series of $\tilde{k}_a(\exp(i\theta))$ contains only terms with non-negative indices and so $\tilde{k}_a$ can be extended to an analytic function, still denoted by $\tilde{k}_a$, on $|w|<1$. We now define a function k_a on the strip $0<\operatorname{Re} z<1$ by the formula $\tilde{k}_a\circ\mu=k_a$. Then the non-tangential limits of k_a at the line $\operatorname{Re} z=j$ coincide (almost everywhere) with $d\langle g_j(\tau), a\rangle/d\tau$, $(j=0,1)$. Moreover, k_a depends linearly on a. Furthermore,

$$|k_a(z)|\leqslant\max\left\{\sup_\tau\left|\frac{d}{d\tau}\langle g_0(\tau), a\rangle\right|, \sup_\tau\left|\frac{d}{d\tau}\langle g_1(\tau), a\rangle\right|\right\}$$
$$\leqslant\max\{\|g_0\|_{A_0}\|a\|_{A_0}, \|g_1\|_{A_1}\|a\|_{A_1}\}\leqslant\max\{\|g_0\|_{A_0}, \|g_1\|_{A_1}\}\,\|a\|_{\Delta(\bar{A})}.$$

Thus

$$|k_a(z)|\leqslant\|l\|_{A_{[\theta]}'}.$$

Define now the function k by $\langle k(z), a\rangle=k_a(z)\ (z\in S_0)$. Obviously $k(z)\in\Delta(\bar{A})'=\Sigma(\bar{A}')$ (see Theorem 2.7.1), and k is analytic and bounded in S_0. Integrating:

$$g(z)=\int_{1/2}^{z}k(z')\,dz'$$

along a path entirely in S_0, we get a function g with values in $\Sigma(\bar{A}')$, which is analytic on S_0. Also, since its derivative k is bounded, g has a continuous extension to S. Moreover, passing to the limit non-tangentially, we obtain

$$\langle g(j+i\tau+ih)-g(j+i\tau), a\rangle=i\langle g_j(\tau+h)-g_j(\tau), a\rangle \quad (j=0,1).$$

By the density assumptions, we have

$$g(j+i\tau+ih)-g(j+i\tau)=i(g_j(\tau+h)-g_j(\tau))\in A_j' \quad (j=0,1).$$

Furthermore, $g\in\mathscr{G}(\bar{A}')$ and $\|g\|_{\mathscr{G}}=\max(\|g_0\|_{\Lambda(A_0')}, \|g_1\|_{\Lambda(A_1')})$. But for any $g\in\mathscr{G}(\bar{A}')$ and $f\in\mathscr{F}(\bar{A})$

$$l(a)=\int\langle dg(i\tau), P_0(\theta,\tau)f(i\tau)\rangle d\tau+\int\langle dg(1+i\tau), P_1(\theta,\tau)f(1+i\tau)\rangle d\tau$$
$$=\langle g'(\theta), f(\theta)\rangle,$$

because this is true for the generators of $\mathscr{F}_0(\bar{A})$ (Lemma 4.5.3, Formula (2)), and $\mathscr{F}_0(\bar{A})$ is dense in $\mathscr{F}(\bar{A})$). Clearly $a'=g'(\theta)\in\bar{A}'^{[\theta]}$, and thus

$$l(a)=\langle g'(\theta), f(\theta)\rangle=\langle a', a\rangle$$

if $f \in \mathscr{F}(\bar{A})$ with $f(\theta)=a$. Also,

$$\|a'\|_{\bar{A}'^{[\theta]}} \leqslant \|g\|_{\mathscr{G}} = \max(\|g_0\|_{\Lambda(A_0')}, \|g_1\|_{\Lambda(A_1')}) \leqslant \|l\|_{\bar{A}'_{[\theta]}}. \quad \square$$

Proof of Lemma 4.5.3: First, assume that $g \in \Lambda(A')$. We have to prove that $\int \langle dg(x), f(x) \rangle$ is meaningful, and defines a linear functional on $L_1(A)$ with norm at most $\|g\|_{\Lambda(A')}$. By a density argument, it is clearly sufficient to consider continuous functions f with compact support. The integral can then be interpreted as the limit of Riemann sums

$$\sum_j \langle g(t_{j+1}) - g(t_j), f(\tau_j) \rangle \qquad (t_j \leqslant \tau_j \leqslant t_{j+1}).$$

The absolute value of each term is bounded by $\|g\|_{\Lambda(A')} \|f(\tau_j)\|_A (t_{j+1} - t_j)$. Thus we obtain

$$|\int \langle dg(x), f(x) \rangle| \leqslant \|g\|_{\Lambda(A')} \|f\|_{L_1(A)},$$

which is the desired estimate.

Conversely, let $l \in L_1(A)'$, i.e.

$$|l(f)| \leqslant \|l\|_{L_1(A)'} \|f\|_{L_1(A)}.$$

Clearly, with χ_I as the characteristic function of the interval I, $l(\chi_{(0,t)}a) = \langle g(t), a \rangle$ defines an A'-valued function g for $t \geqslant 0$ and analogously $-l(\chi_{(t,0)}a) = \langle g(t), a \rangle$ for $t < 0$. It follows that

$$\|g(s) - g(t)\|_{A'} = \sup_{\|a\|_A = 1} |\langle g(s) - g(t), a \rangle| \leqslant |s - t| \, \|l\|_{L_1(A)'}$$

and thus $\|g\|_{\Lambda(A')} \leqslant \|l\|_{L_1(A)'}$. Moreover, we may write, with a Stieltjes integral,

$$l(\chi_I a) = \int_I (d/dx) \langle g(x), a \rangle \, dx = \int \langle dg(x), \chi_I(x) \cdot a \rangle$$

for any bounded interval I. The linear hull of functions of type $\chi_I a$ being dense in $L_1(A)$, we conclude that this representation is valid also for $f \in L_1(A)$. Now (1) and (2) follow. $\square$

4.6. The Reiteration Theorem

Here we show that the complex interpolation method is stable for repeated use in the sense of the theorem below. For its proof, we invoke the equivalence theorem 4.3.1, and the duality theorem 4.5.1.

4.6.1. Theorem (The reiteration theorem). *Let $\bar{A}$ be a compatible Banach couple and put*

$$X_j = \bar{A}_{[\theta_j]} \qquad (0 \leqslant \theta_j \leqslant 1; j = 0, 1).$$

Assume that $\Delta(\bar{A})$ *is dense in the spaces* A_0, A_1 *and* $\Delta(\bar{X})$. *Then*

$$\bar{X}_{[\theta]} = \bar{A}_{[\theta]} \quad (0 \leqslant \eta \leqslant 1;\ \textit{equal norms}),$$

where $\theta = (1-\eta)\theta_0 + \eta\theta_1$. *Note that* $\Delta(\bar{A})$ *is dense in* $\Delta(\bar{X})$ *if* $A_0 \subset A_1$.

Proof: First we show that $\|a\|_{\bar{X}_{[\eta]}} \leqslant \|a\|_{\bar{A}_{[\theta]}}$ if $a \in \bar{A}_{[\theta]}$. Take $a \in \bar{A}_{[\theta]}$; then there exists a function $f \in \mathscr{F}(\bar{A})$, such that $f(\theta) = a$ and $\|f\|_{\mathscr{F}} \leqslant \|a\|_{\bar{A}_{[\theta]}} + \varepsilon$, $\varepsilon > 0$ being arbitrary. Put $f_1(z) = f((1-z)\theta_0 + z\theta_1)$. Then $f_1(\eta) = a$ and

$$f_1(j+it) = f((1-j)\theta_0 + it(\theta_1 - \theta_0)) \in \bar{A}_{[\theta_j]} = X_j \quad (j = 0, 1),$$
$$f_1(j+it) \to 0, \quad |t| \to \infty.$$

Also

$$\|f_1\|_{\mathscr{F}(\bar{X})} \leqslant \|f\|_{\mathscr{F}(\bar{A})} \leqslant \|a\|_{\bar{A}_{[\theta]}} + \varepsilon.$$

This gives $\|a\|_{\bar{X}_{[\theta]}} \leqslant \|a\|_{\bar{A}_{[\theta]}}$.

Similarly, we have $\|a\|_{\bar{Y}^{[\eta]}} \leqslant \|a\|_{\bar{A}^{[\theta]}}$ if $a \in \bar{A}^{[\theta]}$, where $Y_j = \bar{A}^{[\theta_j]}$ $(j = 0, 1)$. To see this, choose $g \in \mathscr{G}(\bar{A})$ such that $g'(\theta) = a$ and $\|g\|_{\mathscr{G}} \leqslant \|a\|_{\bar{A}^{[\theta]}} + \varepsilon$, $\varepsilon > 0$ arbitrary. Put $g_1(z) = (\theta_1 - \theta_0)^{-1} f((1-z)\theta_0 + z\theta_1)$. It is easily verified that $g_1 \in \mathscr{G}(\bar{Y})$, $g_1'(\eta) = a$ und $\|g_1\|_{\mathscr{G}(\bar{Y})} \leqslant \|g\|_{\mathscr{G}(\bar{A})} \leqslant \|a\|_{\bar{A}^{[\theta]}} + \varepsilon$. Thus it follows that $\|a\|_{\bar{Y}^{[\eta]}} \leqslant \|a\|_{\bar{A}^{[\theta]}}$.

To prove the converse inequality $\|a\|_{\bar{A}_{[\theta]}} \leqslant \|a\|_{\bar{X}_{[\eta]}}$ $(a \in \bar{X}_{[\eta]})$, we shall see that it is enough to prove that $\|l\|_{\bar{A}_{[\theta]}'} \geqslant \|l\|_{\bar{X}_{[\eta]}'}$ $(l \in \bar{A}_{[\theta]}')$. In view of the first part of this proof and the duality theorem it follows that

$$\|l\|_{\bar{A}_{[\theta]}'} = \|l\|_{\bar{A}'^{[\theta]}} \geqslant \|l\|_{(\bar{A}'^{[\theta_0]}, \bar{A}'^{[\theta_1]})^{[\eta]}} = \|l\|_{\bar{X}'^{[\eta]}} = \|l\|_{\bar{X}_{[\eta]}'} \qquad (l \in \bar{A}_{[\theta]}'),$$

since, evidently, $\Delta(\bar{X})$ is dense in X_0 and in X_1.

From the first part of the proof and from the inequality $\|l\|_{\bar{A}_{[\theta]}'} \geqslant \|l\|_{\bar{X}_{[\eta]}'}$ it follows that the norms on $\bar{A}_{[\theta]}$ and $\bar{X}_{[\eta]}$ agree on $\bar{A}_{[\theta]}$. By assumption, $\Delta(\bar{A})$ is dense in $\Delta(\bar{X})$. Since $\Delta(\bar{X})$ is dense in $\bar{X}_{[\eta]}$, we conclude that $\Delta(\bar{A})$ is dense in $\bar{X}_{[\eta]}$ and in $\bar{A}_{[\theta]}$. But then we must have $\bar{X}_{[\eta]} = \bar{A}_{[\theta]}$ with equal norms.

The last observation of the theorem follows from Theorem 4.2.1. □

4.7. On the Connection with the Real Method

Let $\bar{A}$ be a compatible Banach couple. The next two theorems provide connections between the complex and the real interpolation methods.

4.7.1. Theorem. *The following inclusions hold*

$$\bar{A}_{\theta,1} \subset \bar{A}_{[\theta]} \subset \bar{A}_{\theta,\infty}$$

if $0 < \theta < 1$.

Proof: Since C_θ is an interpolation functor of exponent θ, the first part follows at once from Theorem 3.9.1.

The second inclusion follows from the Phragmén-Lindelöf extension of the maximum principle. Let $a \in \bar{A}_{[\theta]}$, i.e. there is a function $f \in \mathscr{F}(\bar{A})$ with $f(\theta)=a$. By the Phragmén-Lindelöf principle, we have, since $K(t, f(j+i\tau); \bar{A}) \leq \|f(j+i\tau)\|_{A_j}$,

$$\|a\|_{\theta,\infty} = \sup_t t^{-\theta} K(t, f(\theta); \bar{A})$$
$$\leq \sup_t t^{-\theta} \sup_\tau \|f(i\tau)\|_{A_0}^{1-\theta} t^\theta \sup_\tau \|f(1+i\tau)\|_{A_1}^\theta \leq \|f\|_{\mathscr{F}}.$$

Taking the infimum, this estimate completes the proof. □

4.7.2. Theorem. *If* $0<\theta_0<\theta_1<1, \theta=(1-\eta)\theta_0+\eta\theta_1$ *and* $0<p\leq\infty$ *then*

$$(\bar{A}_{[\theta_0]}, \bar{A}_{[\theta_1]})_{\eta,p} = \bar{A}_{\theta,p} \quad (\textit{equivalent norms}).$$

If $1 \leq p_i \leq \infty$ $(i=0,1)$ *and* $1/p=(1-\eta)/p_0+\eta/p_1$ *then*

$$(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})_{[\eta]} = \bar{A}_{\theta,p} \quad (\textit{equivalent norms}).$$

Proof: The first assertion follows from Theorem 4.7.1 and the reiteration theorem 3.10.5.

The second assertion we prove in two steps; the first step is the inclusion $\bar{A}_{\theta,p} \subset (\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})_{[\eta]}$. Take $a \in \bar{A}_{\theta,p}$ $(a \neq 0)$. Then there is a decomposition $a=\sum_\nu u_\nu$ (in $\Sigma(\bar{A})$) such that $u_\nu \in \Delta(\bar{A})$ and $(\sum_\nu (2^{-\nu\theta} J(2^\nu, u_\nu; \bar{A}))^p)^{1/p} \leq C\|a\|_{\theta,p}$. Put $(\delta>0; 0 \leq \operatorname{Re} z \leq 1)$

$$f(z) = \exp(\delta(z-\eta)^2) \sum_\nu f_\nu$$

where

$$f_\nu(z) = u_\nu \cdot \{2^{(\theta_1-\theta_0)\nu} (2^{-\theta\nu} J(2^\nu, u_\nu; \bar{A}) / \|a\|_{\bar{A}_{\theta,p}})^{p(1/p_1-1/p_0)}\}^{z-\eta}.$$

We obtain

$$|\exp(-\delta(it-\eta)^2| \, \|f(it)\|_{\bar{A}_{\theta_0,p_0}} \leq (\textstyle\sum_\nu (2^{-\theta_0\nu} J(2^\nu, f_\nu(it); \bar{A})^{p_0})^{1/p_0} \leq C\|a\|_{\bar{A}_{\theta,p}}.$$

Similarly, we have

$$\exp(-\delta(1+it-\eta)^2)| \, \|f(1+it)\|_{\bar{A}_{\theta_1,p_1}} \leq C\|a\|_{\bar{A}_{\theta,p}}.$$

Now $f \in \mathscr{F}(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ and $f(\eta)=a$. Thus the inclusion follows.

Conversely, take $a \in (\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})_{[\eta]}$. Let $f \in \mathscr{F}(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ with $f(\eta)=a$, and put

$$g_\nu(z) = 2^{(z-\eta)((\theta_0-\theta_1)\nu+\gamma)} f(z).$$

Clearly, $g_\nu \in \mathscr{F}(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ and $g_\nu(\eta)=a$. The Cauchy integral formula (in $\Sigma(\bar{A})$)

$$a = \int P_0(\eta,\tau) g_\nu(i\tau)\,d\tau + \int P_1(\eta,\tau) g_\nu(1+i\tau)\,d\tau$$

gives

$$\begin{aligned}2^{-\nu\theta}K(2^\nu,a;\bar{A}) &\leqslant 2^{-\nu\theta-\nu\eta(\theta_0-\theta_1)-\eta\gamma}\int P_0(\eta,\tau)K(2^\nu,f(i\tau);\bar{A})d\tau\\ &\quad+2^{-\nu\theta+\nu(1-\eta)(\theta_0-\theta_1)+(1-\eta)\gamma}\int P_1(\eta,\tau)K(2^\nu,f(1+i\tau);\bar{A})d\tau\\ &\leqslant C(\int P_0(\eta,\tau)2^{-\nu\theta_0}K(2^\nu,f(i\tau);\bar{A})d\tau)^{1-\eta}\\ &\quad\cdot(\int P_1(\eta,\tau)2^{-\nu\theta_1}K(2^\nu,f(1+i\tau);\bar{A})d\tau)^\eta,\end{aligned}$$

if γ is chosen appropriately. Using the inequalities of Hölder and Minkowski, this estimate gives

$$\begin{aligned}\|a\|_{\theta,p} &\leqslant C(\int P_0(\eta,\tau)\|f(i\tau)\|^{p_0}_{\bar{A}_{\theta_0,p_0}}d\tau)^{(1-\eta)/p_0}\\ &\quad\cdot(\int P_1(\eta,\tau)\|f(1+i\tau)\|^{p_1}_{\bar{A}_{\theta_1,p_1}}d\tau)^{\eta/p_1}\leqslant C\,\|f\|_{\mathscr{F}},\end{aligned}$$

which yields the required inequality. □

4.8. Exercises

1. (Calderón [2]). Assume that $\bar{A}$ is a compatible Banach couple and that A_0 and A_1 are Banach algebras with the same multiplication in $\Delta(\bar{A})$. Prove that $\Delta(\bar{A})$ is a subalgebra of A_0 and A_1 and that $\bar{A}_{[\theta]}$ may be made into an algebra with

$$\|ab\|_{\bar{A}_{[\theta]}}\leqslant\|a\|_{\bar{A}_{[\theta]}}\|b\|_{\bar{A}_{[\theta]}}$$

for $a,b\in\bar{A}_{[\theta]}$.

Hint: Apply Theorem 4.5.1.

2. Prove that $\bar{A}_{[\theta]}$ is reflexive when both A_0 and A_1 are reflexive.

3. (Krein-Petunin [1]). Assume that $T\colon A_i\to B_i$ is linear with norm M_i, $i=0,1$, $\bar{A}$ and $\bar{B}$ being compatible Banach couples. Show that if $A\subset\bar{A}_{[\theta]}$ and $B'\subset\bar{B}'_{[\theta]}$ then $T\colon A\to B$ with norm at most $M_0^{1-\theta}M_1^\theta$.

Hint: Consider the function $\langle Tf(z),g(z)\rangle$.

4. (Stafney [1]). Prove that if $\Delta(\bar{A})$ is dense in both A_0 and A_1, and if $\bar{A}_{[\theta]}=A_0$ for some θ $(0<\theta<1)$ then $A_0=A_1$. (Cf. Chapter 3, Exercise 21.)

5. Prove first inclusion of Theorem 4.7.1 directly, i.e. without recourse to Theorem 3.9.1.

4.9. Notes and Comment

The complex interpolation method is based on the main idea in Thorin's proof of Riesz's interpolation theorem (cf. Section 1.1). It was introduced in about 1960 by A. P. Calderón [1] and J.L. Lions [2]. Also, S.G. Krein [1] (see Krein and Yu.I. Petunin [1]) has considered "*an analytic scale of Banach spaces*", which yields the same spaces as the complex method. M. Schechter [1] has made *a generalization of the complex method using certain distributions T instead of δ_θ* as in this chapter. These ideas were also partly considered by Lions [2].

The real methods in Chapter 3 are readily extended to *the quasi-normed case* for the complex methods, no corresponding extension has been made as far as we know. For results in this direction (mainly $L_p, 0<p<1$) see Postylnik [1] and Rivière [1]. The main point seems to be the question of a maximum principle. (Cf. Peetre [28].) Also, to the real method there is a discrete (equivalent) counterpart, but we do not know of any such counterpart to the complex method.

4.9.1.—4.9.6. The theorems and the proofs of the first six sections are, except for superficial changes, taken over from Calderón [2]. Moreover, Calderón's paper contains additional material which is not included here. Applications, however, are given in the next two chapters. Of the results not included, we want to mention this: "$\bar{A}_{[\theta]}, 0<\theta<1$, *is reflexive if (at least) one of A_0 and A_1 is reflexive*".

Let us also point out that several of the exercises in Chapter 1 have counterparts for the abstract method, notably 1.6.11.

4.9.7. The first results connecting the real and the complex method were given by Lions-Peetre [1]. The second part of the proof of Theorem 4.7.1 is based on an idea in Peetre [28]. Note that Theorem 3.9.1 yields another proof of that inclusion under a supplementary density assumption. Theorem 4.7.2 is, in its present form, due to Karadžov [1] and to Bergh.

In general, the real $K_{\theta q}$-method and the complex C_θ-method yield different results. (Cf. Chapter 6.) Moreover, neither of the indices 1 and ∞ in Theorem 4.7.1 can be replaced with a q, $1<q<\infty$. (See Chapter 6, Exercise 23.)

Imposing a restriction on the spaces A_0 and A_1, Peetre [21] was able to demonstrate the inclusion

$$\bar{A}_{\theta,p} \subset \bar{A}_{[\theta]} \quad (0<\theta<1),$$

where p is connected with θ and with the conditions on the spaces A_0 and A_1.

Chapter 5

Interpolation of L_p-Spaces

We investigate the real and complex interpolation of L_p-spaces and Lorentz spaces over a measure space. In particular, we prove a generalized version of the Marcinkiewicz theorem (the Calderón-Marcinkiewicz theorem). We also investigate the real and the complex interpolation spaces between L_p-spaces with different measures, thus extending a theorem by Stein and Weiss. In Section 6, we consider the interpolation of vector-valued L_p-spaces of sequences, thus preparing for the interpolation of Besov spaces in the next chapter.

5.1. Interpolation of L_p-Spaces: the Complex Method

Here we shall use the idea in the proof of the Riesz-Thorin theorem to prove the following result.

5.1.1. Theorem. *Assume that* $p_0 \geqslant 1$, $p_1 \geqslant 1$ *and* $0<\theta<1$. *Then*

$$(L_{p_0}, L_{p_1})_{[\theta]} = L_p \quad (\textit{equal norms}),$$

if

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Proof: It is sufficient to prove that

$$\|a\|_{[\theta]} = \|a\|_{(L_{p_0}, L_{p_1})_{[\theta]}} = \|a\|_{L_p}$$

for all bounded functions a with compact support. Put

$$f(z) = \exp(\varepsilon z^2 - \varepsilon\theta^2)\,|a(x)|^{p/p(z)} a(x)/|a(x)|\,,$$

where $1/p(z) = (1-z)/p_0 + z/p_1$. Assuming that $\|a\|_{L_p}=1$ we have $f \in \mathscr{F}$ and $\|f\|_{\mathscr{F}} \leqslant \exp(\varepsilon)$. Since $f(\theta)=a$ we conclude $\|a\|_{[\theta]} \leqslant \exp(\varepsilon)$, whence $\|a\|_{[\theta]} \leqslant \|a\|_{L_p}$.

The converse inequality follows from the relation

$$\|a\|_{L_p}=\sup\{|\langle a,b\rangle|: \|b\|_{L'_p}=1,\ b \text{ bounded with compact support}\}.$$

In fact, put

$$g(z)=\exp(\varepsilon z^2-\varepsilon\theta^2)\,|b(x)|^{p'/p'(z)}b(x)/|b(x)|,$$

where b is as above and $1/p'(z)=(1-z)/p'_0+z/p'_1$. Writing $F(z)=\langle f(z),g(z)\rangle$, we then have $|F(it)|\leqslant\exp(\varepsilon)$, $|F(1+it)|\leqslant\exp(2\varepsilon)$ provided that $\|a\|_{[\theta]}=1$. Thus, by the three line theorem, it follows that $|\langle a,b\rangle|\leqslant|F(\theta)|\leqslant\exp(2\varepsilon)$. This implies $\|a\|_{L_p}\leqslant\|a\|_{[\theta]}$. □

Since we know that the complex interpolation method is an exact interpolation functor of exponent θ, we will get the Riesz-Thorin interpolation theorem as an immediate corollary of Theorem 5.1.1.

It is possible to extend the previous theorem to vector-valued L_p-spaces. Let A be a Banach space and consider the space $L_p(A)=L_p(U,d\mu;A)$ of all strongly measurable functions f such that

$$\int_U \|f(x)\|_A^p\,d\mu(x)<\infty,$$

where $1\leqslant p<\infty$. We shall denote by $L_\infty(A)=L_\infty(U,d\mu;A)$ the completion in the sup-norm of all functions

$$(3)\qquad s(x)=\textstyle\sum_k a_k\chi_{E_k}(x),\qquad a_k\in A,$$

where the sum is finite and χ_{E_k} is the characteristic function of the measurable disjoint sets E_k. Functions of the form (3) will be called simple functions if in addition $\mu(E_k)<\infty$. The completion in $L_\infty(A)$ of the simple functions is denoted by $L_\infty^0(A)$. Note that if A is the space of complex numbers then $L_\infty(A)$ is the space of essentially bounded functions.

5.1.2. Theorem. *Assume that A_0 and A_1 are Banach spaces and that $1\leqslant p_0<\infty$, $1\leqslant p_1<\infty$, $0<\theta<1$. Then*

$$(L_{p_0}(A_0),L_{p_1}(A_1))_{[\theta]}=L_p((A_0,A_1)_{[\theta]})\quad (\textit{equal norms}),$$

where $1/p=(1-\theta)/p_0+\theta/p_1$. If $1\leqslant p_0<\infty$ we also have

$$(L_{p_0}(A_0),L_\infty^0(A_1))_{[\theta]}=L_p((A_0,A_1)_{[\theta]}),$$

with $1/p=(1-\theta)/p_0$.

Proof: Let S denote the space of simple functions with values in $\Delta(\bar{A})$. S is dense in $L_{p_0}(A_0)\cap L_{p_1}(A_1)$, and thus also in $(L_{p_0}(A_0),L_{p_1}(A_1))_{[\theta]}$ and in $L_p(\bar{A}_{[\theta]})$, by Theorem 4.2.2. From now on we consider only functions in S; this is clearly enough.

First we prove the inequality

$$\|a\|_{(L_{p_0}(A_0), L_{p_1}(A_1))_{[\theta]}} \leqslant \|a\|_{L_p(\bar{A}_{[\theta]})}.$$

Since $a \in S$, there is a function $g(\cdot, x) \in \mathscr{F}(\bar{A})$, such that $\|g(\cdot, x)\|_{\mathscr{F}(\bar{A})} \leqslant (1+\varepsilon)\|a(x)\|_{\bar{A}_{[\theta]}}$ $(x \in U; \varepsilon > 0)$, and with $g(\theta, x) = a(x)$ $(x \in U)$.

Put

$$f(z, x) = g(z, x)(\|a(x)\|_{\bar{A}_{[\theta]}} / \|a\|_{L_p(\bar{A}_{[\theta]})})^{p(1/p_0 - 1/p_1)(z-\theta)}.$$

For this function f, we have

$$\|f(it, \cdot)\|_{L_{p_0}(A_0)} = (\int_U \|f(it, x)\|_{A_0}^{p_0} d\mu(x))^{1/p_0} \leqslant (1+\varepsilon)\|a\|_{L_p(\bar{A}_{[\theta]})},$$

where some elementary calculations have been left to the reader. Similarly,

$$\|f(1+it, \cdot)\|_{L_{p_1}(A_1)} \leqslant (1+\varepsilon)\|a\|_{L_p(\bar{A}_{[\theta]})},$$

and the desired inequality follows, since $\varepsilon > 0$ was arbitrary.

The other inequality follows from Lemma 4.3.2 and Hölder's inequality $(p_0/p(1-\theta) > 1;\ p_1/p\theta > 1)$. In fact, if $f(\cdot, x) \in \mathscr{F}(\bar{A})$ and $f(\theta, x) = a(x)$ $(x \in U)$ then

$$\begin{aligned}\|a\|_{L_p(\bar{A}_{[\theta]})} &= (\int_U \|a(x)\|_{\bar{A}_{[\theta]}}^p d\mu)^{1/p} \\ &\leqslant (\int_U \{(1-\theta)^{-1} \int_{-\infty}^{\infty} \|f(i\tau, x)\|_{A_0} P_0(\theta, \tau)\, d\tau)^{1-\theta} \\ &\quad \cdot (\theta^{-1} \int_{-\infty}^{\infty} \|f(1+i\tau, x)\|_{A_1} P_1(\theta, \tau)\, d\tau)^{\theta}\}^p d\mu)^{1/p} \\ &\leqslant \sup_\tau \|f(i\tau)\|_{L_{p_0}(A_0)}^{1-\theta} \sup_\tau \|f(1+i\tau)\|_{L_{p_1}(A_1)}^{\theta} \\ &\leqslant \|f\|_{\mathscr{F}(L_{p_0}(A_0), L_{p_1}(A_1))}.\end{aligned}$$

This gives the conclusion.

The statement about L_∞^0 is proved in precisely the same manner. □

5.2. Interpolation of L_p-Spaces: the Real Method

In this section, L_p will denote the space $L_p(U, d\mu; A) = L_p(d\mu; A)$, consisting of all strongly μ-measurable functions with values in the Banach space A which satisfy

$$\|f\|_p^p = \int_U \|f(x)\|_A^p d\mu < \infty.$$

To simplify formulas etc., all statements and their proofs are given for the case $A = \mathbb{C}$. However, it is not hard to see that the results hold also in the general case. In this and the next section, we consider interpolation only of $L_p(d\mu; A)$-spaces with A and $d\mu$ fixed.

We shall identify the space $(L_{p_0}, L_{p_1})_{\theta,q}$ with a Lorentz space, employing an explicit formula for $K(t, f; L_p, L_\infty)$ and the reiteration theorem.

The reader is asked to recall the definitions of the decreasing rearrangement f^* of a function f, and of the Lorentz space $L_{p,q}$, as they are given in Section 1.3. Note that these definitions have a sense also in the vector-valued case.

5.2.1. Theorem. *Suppose that* $f \in L_p + L_\infty$, $0<p<\infty$. *Then*

(1) $$K(t, f; L_p, L_\infty) \sim \left(\int_0^{t^p} (f^*(s))^p ds\right)^{1/p}.$$

If $p=1$ *there is equality in* (1).

Moreover, with $0<p_0<p_1 \leqslant \infty$,

(2) $$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p,q} \quad (\textit{equivalent norms})$$

if $p_0 < q \leqslant \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$ $(0<\theta<1)$. *In particular,* $(L_{p_0}, L_{p_1})_{\theta,p} = L_p$.

Proof: First we prove "$\leqslant$" of (1). Take

$$f_0(x) = \begin{cases} f(x) - f^*(t^p) f(x)/|f(x)| & \textit{if} \quad |f(x)| > f^*(t^p) \\ 0 \quad \textit{otherwise} \end{cases}$$

and $f_1 = f - f_0$. Let $E = \{x \mid f_0(x) \neq 0\}$. Then $\mu(E) \leqslant t^p$, and we have, since $f^*(s)$ is constant on $[\mu(E), t^p]$,

$$\begin{aligned} K(t, f; L_p, L_\infty) &\leqslant \|f_0\|_p + t\|f_1\|_\infty \\ &= \left(\int_E (|f(x)| - f^*(t^p))^p d\mu\right)^{1/p} + t f^*(t^p) \\ &= \left(\int_0^{\mu(E)} (f^*(s) - f^*(t^p))^p ds\right)^{1/p} + \left(\int_0^{t^p} (f^*(t^p))^p ds\right)^{1/p} \\ &= \left(\int_0^{t^p} (f^*(s) - f^*(t^p))^p ds\right)^{1/p} + \left(\int_0^{t^p} (f^*(t^p))^p ds\right)^{1/p} \\ &\leqslant C\left(\int_0^{t^p} (f^*(s))^p ds\right)^{1/p}, \end{aligned}$$

where $C=1$ if $p=1$. For the converse inequality, assume that $f = f_0 + f_1$, $f_0 \in L_p$, $f_1 \in L_\infty$. Using the inequality $m(\sigma_0 + \sigma_1, f) \leqslant m(\sigma_0, f_0) + m(\sigma_1, f_1)$, we obtain, by elementary calculations,

$$f^*(s) \leqslant f_0^*((1-\varepsilon)s) + f_1^*(\varepsilon s), \qquad 0<\varepsilon<1.$$

Thus

$$\begin{aligned} \left(\int_0^{t^p} (f^*(s))^p ds\right)^{1/p} &\leqslant C\left\{\left(\int_0^{t^p} (f_0^*((1-\varepsilon)s))^p ds\right)^{1/p} + \left(\int_0^{t^p} (f_1^*(\varepsilon s))^p ds\right)^{1/p}\right\} \\ &\leqslant C\left\{\left(\int_0^\infty (f_0^*((1-\varepsilon)s))^p ds\right)^{1/p} + t f_1^*(0)\right\} \\ &= C\{(1-\varepsilon)^{-1/p} \|f_0\|_p + t\|f_1\|_\infty\}. \end{aligned}$$

Taking the infimum and letting $\varepsilon \to 0$, we get (1). Note that $C=1$ if $p \geqslant 1$.

In order to prove (2), we first establish (2) for $p_1=\infty$ and then we apply the reiteration theorem 3.5.3. Thus let $p_1=\infty$. By Formula (1), we have

$$\begin{aligned}\|f\|_{(L_{p_0}, L_\infty)_{\theta,q}} &= \left(\int_0^\infty (t^{-\theta} K(t,f; L_{p_0}, L_\infty))^q \, dt/t\right)^{1/q} \\ &\sim \left(\int_0^\infty \left(t^{-\theta p_0} \int_0^{t^{p_0}} (f^*(s))^{p_0} ds\right)^{q/p_0} dt/t\right)^{1/q} \\ &= \left(\int_0^\infty \left(t^{-\theta p_0 + p_0} \int_0^1 (f^*(s t^{p_0}))^{p_0} s \, ds/s\right)^{q/p_0} dt/t\right)^{p_0/q}.\end{aligned}$$

Then, by the Minkowski inequality ($q/p_0 > 1$), we get

$$\|f\|_{(L_{p_0}, L_\infty)_{\theta,q}} \leqslant C \int_0^1 \left(s^{q/p_0} \int_0^\infty t^{(1-\theta)q} (f^*(s t^{p_0}))^q \, dt/t\right)^{p_0/q} ds/s \leqslant C \|f\|_{L_{p,q}},$$

since $1/p=(1-\theta)/p_0$. Conversely, because f^* is non-negative and decreasing, it follows that

$$\|f\|_{(L_{p_0}, L_\infty)_{\theta,q}} \geqslant C\left(\int_0^\infty \left(t^{-\theta p_0} t^{p_0} (f^*(t^{p_0}))^{p_0}\right)^{q/p_0} dt/t\right)^{1/q} \geqslant C \|f\|_{L_{p,q}}.$$

Thus, (2) is established for $p_1=\infty$. From this and the reiteration theorem 3.5.3, we infer that ($p_1<\infty$)

$$\begin{aligned}(L_{p_0}, L_{p_1})_{\theta,q} &= ((L_r, L_\infty)_{\theta_0, p_0}, (L_r, L_\infty)_{\theta_1, p_1})_{\theta,q} \\ &= (L_r, L_\infty)_{\eta,q} = L_{p,q} \quad (\textit{equivalent norms}),\end{aligned}$$

where $0<r<p_0$, and θ_0, θ_1, η have their prescribed values. ▯

Note that there is another proof of the last statement of Theorem 5.2.1 in the case $p_1<\infty$, using the power theorem. In fact, we have the following result.

5.2.2. Theorem. *If* $0<p_0<p_1<\infty$ *and* $p=(1-\eta)p_0+\eta p_1$, $0<\eta<1$ *then*

$$((L_{p_0})^{p_0}, (L_{p_1})^{p_1})_{\eta,1} = (L_p)^p.$$

The quasi-norm on $(L_p)^p$ *is a constant multiplied by the quasi-norm on* $((L_{p_0})^{p_0}, (L_{p_1})^{p_1})_{\eta,1}$.

Using Theorem 5.2.2 combined with the power theorem 3.11.6 we conclude that $L_p=(L_{p_0}, L_{p_1})_{\theta,p}$ with equivalence of quasi-norms.

Proof: We may assume that $p_0<p_1$. Let us write

$$L(t,f) = K(t,f; (L_{p_0})^{p_0}, (L_{p_1})^{p_1}).$$

Then

$$\begin{aligned}L(t,f) &= \inf_{f=f_0+f_1} \int_U (|f_0(x)|^{p_0} + t|f_1(x)|^{p_1})\, d\mu \\ &= \int_U \inf_{f(x)=f_0(x)+f_1(x)} (|f_0(x)|^{p_0} + t|f_1(x)|^{p_1})\, d\mu.\end{aligned}$$

But

$$\inf_{y=y_0+y_1}(|y_0|^{p_0}+t|y_1|^{p_1})=|y|^{p_0}F(t|y|^{p_1-p_0})$$

where $F(s)=\inf_{y_0+y_1=1}(|y_0|^{p_0}+s|y_1|^{p_1})\sim\min(1,s)$. Then

$$L(t,f)=\int_U|f(x)|^{p_0}F(t|f(x)|^{p_1-p_0})\,d\mu\,.$$

It follows that

$$\begin{aligned}\|f\|_{((L_{p_0})^{p_0},(L_{p_1})^{p_1})_{\eta,1}} &= \int_0^\infty t^{-\eta}L(t,f)\,dt/t\\ &= \int_U|f(x)|^{p_0}\int_0^\infty t^{-\eta}F(t|f(x)|^{p_1-p_0})(dt/t)\,d\mu\,.\end{aligned}$$

Writing

$$c_0=\int_0^\infty t^{-\eta}F(t)\,dt/t$$

we obtain

$$\|f\|_{((L_{p_0})^{p_0},(L_{p_1})^{p_1})_{\eta,1}}=c_0\int_U|f(x)|^p d\mu=c_0\|f\|_{L_p}^p\,. \quad \square$$

Using Theorem 5.2.1 and 5.2.2, we can now prove the following version of the Riesz-Thorin interpolation theorem.

5.2.3. Theorem. *Write* $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$ *where* $0<\theta<1$ *and* $0<p_i,q_i\leqslant\infty$, $i=0,1$. *Assume that* $p\leqslant q$. *Then*

$$T\colon L_{p_0}(U,d\mu)\to L_{q_0}(V,d\nu)\,,$$
$$T\colon L_{p_1}(U,d\mu)\to L_{q_1}(V,d\nu)\,,$$

implies that

$$T\colon L_p(U,d\mu)\to L_q(V,d\nu)\,.$$

If M_i *is the quasi-norm of* $T\colon L_{p_i}\to L_{q_i}$ *and if* M *is the quasi-norm of* $T\colon L_p\to L_q$, *then*

$$M\leqslant CM_0^{1-\theta}M_1^\theta\,.$$

If $p_i=q_i<\infty$ $(i=0,1)$ *then* $C=1$.

Note that the theorem holds in the quasi-normed case, but that we have the restriction $p\leqslant q$.

Proof: Assume first that $p_i\neq q_i$ for some $i\in\{0,1\}$. By Theorem 5.2.1, Theorem 3.4.1 and the interpolation property, we have

$$\begin{aligned}\|Tf\|_q &\leqslant C\|Tf\|_{(L_{q_0},L_{q_1})_{\theta,q}}\leqslant CM_0^{1-\theta}M_1^\theta\|f\|_{(L_{p_0},L_{p_1})_{\theta,q}}\\ &\leqslant CM_0^{1-\theta}M_1^\theta\|f\|_{(L_{p_0},L_{p_1})_{\theta,p}}\leqslant CM_0^{1-\theta}M_1^\theta\|f\|_p\,,\end{aligned}$$

where C depends on θ.

If $p_i = q_i$, $i = 0, 1$, we use Theorem 3.11.8 to obtain

$$\|Tf\|_{((L_{p_0})^{p_0}, (L_{p_1})^{p_1})_{\eta,1}} \leq M_0^{(1-\eta)p_0} M_1^{\eta p_1} \|f\|_{((L_{p_0})^{p_0}, (L_{p_1})^{p_1})_{\eta,1}} .$$

With $(1-\eta)p_0 = (1-\theta)p$ and $\eta p_1 = \theta p$ we obtain, by Theorem 5.2.1,

$$c_0 \|Tf\|_p^p \leq (M_0^{1-\theta} M_1^{\theta})^p c_0 \|f\|_p^p,$$

which gives the result. □

We conclude this section with a proof of the following complement to the reiteration theorem. (This is an extension of Theorem 3.5.4.)

5.2.4. Theorem. *Assume that A_0 and A_1 are Banach spaces. Then for $0 < q_0 \leq \infty$, $0 < q_1 \leq \infty$ we have*

$$(\bar{A}_{\theta,q_0}, \bar{A}_{\theta,q_1})_{\eta,q} = \bar{A}_{\theta,q}$$

where $0 < \eta < 1$ and

$$\frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1} .$$

Proof: Using Theorem 5.2.1 we see that

$$(3) \qquad (\lambda^{\theta,q_0}, \lambda^{\theta,q_1})_{\eta,q} = \lambda^{\theta,q} .$$

(The case $q_0 = q_1 \geq 1$ is a consequence of Theorem 3.4.1.) Write $X_j = \bar{A}_{\theta,q_j}$. We shall prove that

$$(4) \qquad \|a\|_{\bar{X}_{\eta,q}} \sim \|(K(2^\nu, a; \bar{A}))_\nu\|_{(\lambda^{\theta,q_0}, \lambda^{\theta,q_1})_{\eta,q}} .$$

Clearly this implies the result, since (3) implies

$$\|a\|_{\bar{X}_{\eta,q}} \sim \|(K(2^\nu, a; \bar{A}))_\nu\|_{\lambda^{\theta,q}} \sim \|a\|_{\bar{A}_{\theta,q}} .$$

(See Lemma 3.1.3.)

In order to prove (4) we assume first that $a \in \bar{X}_{\eta,q}$. Let $a = \sum_\mu u_\mu$ with

$$\|(J(2^\mu, u_\mu; \bar{X}))_\mu\|_{\lambda^{\eta,q}} \leq C \|a\|_{\bar{X}_{\eta,q}} .$$

Then

$$\begin{aligned} &\|(K(2^\nu, a; \bar{A}))_\nu\|_{(\lambda^{\theta,q_0}, \lambda^{\theta,q_1})_{\eta,q}} \\ &\quad \leq \|(\textstyle\sum_\mu K(2^\nu, u_\mu; \bar{A}))_\nu\|_{(\lambda^{\theta,q_0}, \lambda^{\theta,q_1})_{\eta,q}} \\ &\quad \leq C(\textstyle\sum_\mu (2^{-\mu\eta} J(2^\mu, (K(2^\nu, u_\mu; \bar{A}))_\nu; \lambda^{\theta,q_0}, \lambda^{\theta,q_1}))^q)^{1/q} \\ &\quad = C(\textstyle\sum_\mu (2^{-\mu\eta} J(2^\mu, u_\mu; \bar{X}))^q)^{1/q} \leq C \|a\|_{\bar{X}_{\eta,q}} . \end{aligned}$$

Conversely, assume that $(K(2^\nu, a; \bar{A}))_\nu \in (\lambda^{\theta,q_0}, \lambda^{\theta,q_1})_{\eta,q}$. Choose, using the fundamental lemma 3.3.2, $a = \sum_\nu u_\nu$ such that $J(2^\nu, u_\nu; \bar{A}) \leqslant CK(2^\nu, a; \bar{A})$. We shall prove

$$(5) \qquad K(t, a; \bar{X}) \leqslant CK(t, (J(2^\nu, u_\nu; \bar{A}))_\nu; \lambda^{\theta,q_0}, \lambda^{\theta,q_1}).$$

Obviously, this implies the desired inequality:

$$\|a\|_{\bar{X}_{\eta,q}} \leqslant C\|(K(2^\nu, a; \bar{A}))_\nu\|_{(\lambda^{\theta,q_0}, \lambda^{\theta,q_1})_{\eta,q}}.$$

To prove (5), let $J(2^\nu, u_\nu; \bar{A}) = \alpha_{0\nu} + \alpha_{1\nu}$, $\alpha_{i\nu} \in \lambda^{\theta,q_i}$. Put $a_i = \sum_\nu (J(2^\nu, u_\nu; \bar{A}))^{-1}\alpha_{i\nu}u_\nu$. Then $a_0 + a_1 = a$ and

$$\begin{aligned}\|a_i\|_{X_i} &\leqslant C(\textstyle\sum_\nu (2^{-\nu\theta} J(2^\nu, (J(2^\nu, u_\nu; \bar{A}))^{-1}\alpha_{i\nu}u_\nu; \bar{A}))^{q_i})^{1/q_i} \\ &\leqslant C(\textstyle\sum_\nu (2^{-\nu\theta}|\alpha_{i\nu}|)^{q_i})^{1/q_i} = C\|(\alpha_{i\nu})_\nu\|_{\lambda^{\theta,q_i}}.\end{aligned}$$

This clearly proves (5). □

5.3. Interpolation of Lorentz Spaces

In this section we shall characterize the space $(L_{p_0q_0}, L_{p_1q_1})_{\theta,q}$, and then we shall prove a generalization of Marcinkiewicz's interpolation theorem (Theorem 1.3.1).

5.3.1. Theorem. *Suppose that p_0, p_1, q_0, q_1, and q are positive, possibly infinite, numbers and write $1/p = (1-\eta)/p_0 + \eta/p_1$ where $0 < \eta < 1$. Then, if $p_0 \neq p_1$,*

$$(1) \qquad (L_{p_0q_0}, L_{p_1q_1})_{\eta,q} = L_{pq}.$$

This formula is also true in the case $p_0 = p_1 = p$, provided that $1/q = (1-\eta)/q_0 + \eta/q_1$.

Proof: In the case $p_0 \neq p_1$ we use Theorem 5.2.1 and the reiteration theorem 3.5.3. With $0 < r < \min(p_0, p_1)$ and $1/p_i = (1-\theta_i)/r$, $\theta = (1-\eta)\theta_0 + \eta\theta_1$ we obtain, noting that $1/p = (1-\theta)/r$,

$$(L_{p_0q_0}, L_{p_1q_1})_{\eta,q} = ((L_r, L_\infty)_{\theta_0,q_0}, (L_r, L_\infty)_{\theta_1,q_1})_{\eta,q} = (L_r, L_\infty)_{\theta,q} = L_{pq}.$$

In the case $p_0 = p_1 = p$, we use Theorem 3.5.5 instead. □

As a consequence of Theorem 5.3.1 we have the following interpolation theorem, which contains the Marcinkiewicz interpolation theorem.

5.3.2. Theorem (The general Marcinkiewicz interpolation theorem). *Suppose that*

$$T\colon L_{p_0r_0}(U, d\mu) \to L_{q_0s_0}(V, dv),$$
$$T\colon L_{p_1r_1}(U, d\mu) \to L_{q_1s_1}(V, dv),$$

where $p_0 \neq p_1$ *and* $q_0 \neq q_1$. *Put* $1/p=(1-\theta)/p_0+\theta/p_1$ *and* $1/q=(1-\theta)/q_0+\theta/q_1$. *Then*

(4) $$T: L_{pr}(U, d\mu) \to L_{qr}(V, dv), \quad 0<r\leqslant\infty.$$

In particular, we have

(5) $$T: L_p(U, d\mu) \to L_q(V, dv),$$

provided that $p \leqslant q$.

Note that the theorem holds also in the vector-valued case, i.e. when all spaces are spaces of functions with values in a fixed Banach space.

Proof: The conclusion (4) follows at once from Theorem 5.3.1 and (5) then follows from the inclusion $L_{qp} \subset L_{qq}$. □

The most general consequence of Theorem 5.3.1 is that if

$$T: L_{p_i r_i} \to L_{q_i s_i}, \quad i=0,1$$

then

$$T: L_{pr} \to L_{qs}$$

provided that $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$, $p_0 \neq p_1$, $q_0 \neq q_1$ and that $0<r\leqslant s\leqslant\infty$. A particular case is the following result.

5.3.4. Theorem (Calderón's interpolation theorem). *Suppose that* $(\rho>0)$

$$T: L_{p_i \rho} \to L_{q_i \infty}, \quad i=0,1.$$

Then

$$T: L_{pr} \to L_{qs}$$

if $r \leqslant s$ *and if* p_i, q_i, p *and* q *satisfy the assumptions of Theorem 5.3.2.*

5.4. Interpolation of L_p-Spaces with Change of Measure: $p_0 = p_1$

In the preceding sections we considered interpolation of L_p-spaces with a fixed measure μ and varying values of p. Here we shall let μ vary but keep p fixed. In the next section we shall let both μ and p vary.

We shall characterize the space $(L_p(d\mu_0), L_p(d\mu_1))_{\theta,q}$, where μ_0 and μ_1 are two positive measures. We may assume that μ_0 and μ_1 are absolutely continuous with respect to a third measure μ. Thus we suppose that

$$d\mu_0(x)=w_0(x)\,d\mu(x)\,,$$
$$d\mu_1(x)=w_1(x)\,d\mu(x)\,.$$

Let us write

$$L_p(w)=L_p(U,\,w d\mu)\,.$$

5.4.1. Theorem (The interpolation theorem of Stein-Weiss). *Assume that* $0<p\leqslant\infty$ *and that* $0<\theta<1$. *Put*

$$w(x)=w_0^{1-\theta}(x)\,w_1^{\theta}(x)\,.$$

Then

$$(L_p(w_0),\,L_p(w_1))_{\theta,p}=L_p(w)$$

(with equivalent norms). Moreover, if

$$T\colon L_p(U,\,w_0 d\mu)\to L_p(V,\,\tilde{w}_0 dv)\,,$$
$$T\colon L_p(U,\,w_1 d\mu)\to L_p(V,\,\tilde{w}_1 dv)$$

with quasi-norms M_0 *and* M_1 *respectively, then*

$$T\colon L_p(U,\,w d\mu)\to L_p(V,\,\tilde{w} dv)$$

with quasi-norm

$$M\leqslant M_0^{1-\theta}M_1^{\theta}\,.$$

Here $\tilde{w}(x)=\tilde{w}_0^{1-\theta}(x)\,\tilde{w}_1^{\theta}(x)$.

Proof: We shall consider the functional

$$K_p(t,f)=\inf_{f=f_0+f_1}(\|f_0\|^p_{L_p(w_0)}+t^p\|f_1\|^p_{L_p(w_1)})^{1/p}\qquad(0<p\leqslant\infty)\,.$$

Let us write

$$\|f\|_{\theta,q;\,p}=\Phi_{\theta,q}(K_p(t,f))\,.$$

Then we have (Exercise 1)

$$\|f\|_{\theta,q;\,p}\sim\|f\|_{\theta,q}\,.$$

Moreover, since obviously

(1) $$K_p(t,\,Tf)\leqslant M_0K_p(M_1t/M_0,f)\,,$$

we have

$$\|Tf\|_{\theta,q;p} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{\theta,q;p}.$$

Therefore the theorem will follow if we can prove that

$$\|f\|_{\theta,p;p} = c\|f\|_{L_p(w)}. \tag{2}$$

In order to prove (2), we shall prove that

$$K_p(t,f) = \|f\|_{L_p(w_t)}, \tag{3}$$

where

$$w_t(x) = w_0(x) F(t^p w_1(x)/w_0(x)).$$

Here F is defined as in the proof of Theorem 5.2.2, i.e.

$$F(s) = \inf_{y_0+y_1=1}(|y_0|^p + s|y_1|^p).$$

Indeed, if (3) holds, we conclude that

$$\begin{aligned}\|f\|_{\theta,p;p} &= \left(\int_0^\infty t^{-\theta p} \int_U |f(x)|^p w_0(x) F(t^p w_1(x)/w_0(x))\, d\mu(x)\, dt/t\right)^{1/p}\\ &= \left(\int_U |f(x)|^p \int_0^\infty t^{-\theta p} w_0(x) F(t^p w_1(x)/w_0(x))\, d\mu(x)\, dt/t\right)^{1/p}.\end{aligned}$$

Now the last integral is equal to $c^p w_0^{1-\theta} w_1^{\theta}$ where

$$c = \left(\int_0^\infty s^{-\theta p} F(s^p)\, ds/s\right)^{1/p}.$$

Note that $c < \infty$, since $F(s) \sim \min(1,s)$. This gives (2).

The proof of (3) is quite similar to the proof of Theorem 5.2.2. In fact,

$$\begin{aligned}K_p(t,f) &= \left(\inf_{f=f_0+f_1} \int_U (|f_0|^p w_0 + t^p |f_1|^p w_1)\, d\mu\right)^{1/p}\\ &= \left(\int_U (\inf_{f=f_0+f_1} (|f_0|^p w_0 + t^p |f_1|^p w_1)\, d\mu\right)^{1/p}.\end{aligned}$$

Since $\inf_{y_0+y_1=y}(|y_0|^p w_0 + t^p |y_1|^p w_1) = |y|^p w_0 F(t^p w_1/w_0)$, we obtain (3). □

The rest of this section is devoted to the problem of finding all interpolation functions in the sense of our next definition.

5.4.2. Definition. *The positive function h is called an interpolation function of power p if*

$$T\colon (L_p(w_0), L_p(w_1)) \to (L_p(\tilde{w}_0), L_p(\tilde{w}_1))$$

with quasi-norms (M_0, M_1), *implies*

$$T\colon L_p(w_0 h(w_1/w_0)) \to L_p(\tilde{w}_0 h(\tilde{w}_1/\tilde{w}_0))$$

with quasi-norm

$$M \leqslant C\max(M_0, M_1).$$

Clearly $h(t)=t^\theta$ is an interpolation function of power p. This follows from the previous theorem.

Let us assume that h is an interpolation function of power p. Choose $U=\{0,1\}$ and let μ be the measure $\delta_0+\delta_1$ (carrying the mass 1 at each of the two points). Put $w_0(0)=w_0(1)=1$, $w_1(0)=s$, $w_1(1)=t$ and let T be defined by

$$(Ta)(0)=0, \quad (Ta)(1)=a(0).$$

In this case

$$M_0=1, \quad M_1=(t/s)^{1/p}, \quad M=(h(t)/h(s))^{1/p}$$

and thus it follows that

$$h(t) \leqslant C\max(1, t/s)h(s). \tag{4}$$

We call a function h *quasi-concave* if it is equivalent to a concave function, i.e. $h(t)\sim k(t)$ for some concave function k.

5.4.3. Lemma. *Let h be a positive function. Then the following three conditions are equivalent:*
(i) *h is quasi-concave;*
(ii) $h(t)\sim\alpha+\beta t+\int_0^\infty \min(\tau,t)\,dm_0(\tau)$, *where* $\alpha\geqslant 0$, $\beta\geqslant 0$, *and m_0 is an increasing function bounded from above and with* $\lim_{t\to 0} t m_0(t)=0$;
(iii) *h satisfies* (4).

Proof: We prove the following implications: (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i).

Now (ii)$\Rightarrow$(iii) is obvious, since the right hand side in (ii) clearly is concave.

To prove (i)$\Rightarrow$(ii), we assume that $h(t)\sim k(t)$ with k concave. We shall show that we may write

$$k(t)=\alpha+\beta t+\int_0^\infty \min(\tau,t)\,dm_0(\tau),$$

with a suitable choice of α, β, m_0. Take $\alpha=\lim_{t\to 0}k(t)$ and $\beta=\lim_{t\to 0}k(t)/t$. Then the function $k(t)-\alpha-\beta t$ is obviously also positive and concave. Moreover, it follows by partial integration that

$$\begin{aligned} k(t)-\alpha-\beta t &= \int_0^t (k'(\tau)-\beta)\,d\tau = t(k'(t)-\beta)-\int_0^t \tau\, d(k'(\tau)) \\ &= \int_0^\infty \min(\tau,t)\,d(-k'(\tau)), \end{aligned}$$

since k' is non-negative and decreasing, $0\leqslant t(k'(t)-\beta)\leqslant k(t)-\alpha-\beta t\to 0$ as $t\to 0$. Taking $m_0(t)=-k'(t)$, we have proved (i)$\Rightarrow$(ii).

For the remaining implication (iii)⇒(i), we assume that h satisfies (4). Define the function k by

$$k(t)=\sup\{\sum_{i=1}^n \lambda_i h(t_i) \mid \sum_{i=1}^n \lambda_i=1,\ \sum_{i=1}^n \lambda_i t_i=t,\ \lambda_i\geqslant 0\}.$$

Clearly, k is concave and $h(t)\leqslant k(t)$. Conversely, by (4), we obtain

$$\begin{aligned}\sum_i \lambda_i h(t_i) &\leqslant C\sum_i \lambda_i \max(1, t_i/t)\, h(t)\\ &\leqslant C(\sum_i \lambda_i + t^{-1}\sum_i \lambda_i t_i)\, h(t)\leqslant C h(t).\end{aligned}$$

Thus $h(t)\sim k(t)$, with k concave, i.e. h is quasi-concave. □

5.4.4. Theorem. *A positive function h is an interpolation function of power p if and only if it is quasi-concave. In particular, if h is an interpolation function of power p for some p, the same is true for all p.*

Proof: It remains to prove the sufficiency. Let us introduce the function Φ, defined by

$$\Phi(\varphi(\tau))=\left(\alpha \lim_{\tau\to\infty}\varphi(\tau)^p+\beta \lim_{\tau\to 0}\left(\frac{\varphi(\tau)}{\tau}\right)^p+\int_0^\infty \tau^{-p}\varphi(\tau)^p\, dm_0(\tau^{-p})\right)^{1/p}.$$

The assumption is that h is equivalent to the function given in Lemma 5.4.3. Note that

$$\varphi\leqslant\psi \Rightarrow \Phi(\varphi)\leqslant\Phi(\psi), \tag{5}$$

$$\Phi(\varphi(s\tau))\leqslant \max(1,s)\,\Phi(\varphi(\tau)). \tag{6}$$

We shall now prove that

$$\Phi(K_p(\tau, f;\, L_p(w_0),\, L_p(w_1)))\sim \| f \|_{L_p(w_0 h(w_1/w_0))}. \tag{7}$$

This is easily done if we use Formula (3) (note that $F(s)\sim\min(1,s)$) and Lemma 5.4.3. In fact, we have

$$\begin{aligned}\Phi(K_p(\tau, f)) &\sim \Phi((\int_U |f|^p w_0 \min(1, \tau^p w_1/w_0)\,d\mu)^{1/p})\\ &=\left(\int_U |f|^p w_0\left(\alpha+\beta\frac{w_1}{w_0}+\int_0^\infty \tau^{-p}\min\left(1, \tau^p\frac{w_1}{w_0}\right) dm_0(\tau^{-p})\right) d\mu\right)^{1/p}\\ &\sim (\int_U |f|^p w_0 h(w_1/w_0)\,d\mu)^{1/p}.\end{aligned}$$

Next, we shall prove that

$$\Phi(K_p(\tau, Tf;\, L_p(\tilde w_0),\, L_p(\tilde w_1)))\leqslant C\max(M_0, M_1)\,\Phi(K_p(\tau, f;\, L_p(w_0),\, L_p(w_1))). \tag{8}$$

This follows at once from (5), (6) and (1), since

$$\Phi(K_p(\tau, Tf)) \leqslant M_0 \Phi(K_p(M_1\tau/M_0, f)) \leqslant M_0 \max(1, M_1/M_0)\Phi(K_p(\tau, f)).$$

From (7) and (8) we conclude that

$$\|Tf\|_{L_p(\tilde{w}_0 h(\tilde{w}_1/\tilde{w}_0))} \leqslant C \max(M_0, M_1) \|f\|_{L_p(w_0 h(w_1/w_0))}.$$

This completes the proof. □

5.5. Interpolation of L_p-Spaces with Change of Measure: $p_0 \neq p_1$

In this section, we shall investigate the interpolation space $(L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,p}$, where $p_0 \neq p_1$. Here is our result:

5.5.1. Theorem. *Assume that* $0 < p_0 < \infty$ *and* $0 < p_1 < \infty$. *Then we have*

$$(L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,p} = L_p(w), \qquad 0 < \theta < 1,$$

where

$$w = w_0^{p(1-\theta)/p_0} w_1^{p\theta/p_1},$$
$$1/p = (1-\theta)/p_0 + \theta/p_1.$$

Proof: Using the power theorem 3.11.6, we see that

$$((L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,p})^p = ((L_{p_0}(w_0))^{p_0}, (L_{p_1}(w_1))^{p_1})_{\eta,1},$$

where $\eta = \theta p/p_1$. The norm of f in the space on the right hand side is

$$\int_0^\infty t^{-\eta} \inf_{f=f_0+f_1} \int_U (|f_0|^{p_0} w_0 + t|f_1|^{p_1} w_1)\, d\mu\, dt/t$$
$$= \int_U \{\int_0^\infty t^{-\eta} \inf_{f=f_0+f_1} (|f_0|^{p_0} w_0 + t|f_1|^{p_1} w_1)\, dt/t\}\, d\mu.$$

Writing $F(s) = \inf_{y_0+y_1=1}(|y_0|^{p_0} + s|y_1|^{p_1})$, we see that the last integral is equal to

$$\int_U |f|^{p_0} w_0 \{\int_0^\infty t^{-\eta} F(tw_1|f|^{p_1-p_0}/w_0)\, dt/t\}\, d\mu$$
$$= \int_0^\infty s^{-\eta} F(s)\, ds/s \cdot \int_U |f|^{(1-\eta)p_0+\eta p_1} w_0^{1-\eta} w_1^\eta\, d\mu.$$

Since $1-\eta = p(1-\theta)/p_0$ and $(1-\eta)p_0 + \eta p_1 = p$, and since $F(s) \sim \min(1, s)$, we obtain the result. □

As a corollary we get the following extension of the interpolation theorem of Stein-Weiss.

5.5.2. Corollary. *Assume that* $0<p_0<\infty$, $0<p_1<\infty$ *and that*

$$T\colon L_{p_0}(U, w_0 d\mu)\to L_{q_0}(V, \tilde{w}_0 dv),$$
$$T\colon L_{p_1}(U, w_1 d\mu)\to L_{q_1}(V, \tilde{w}_1 dv)$$

with quasi-norms M_0 *and* M_1 *respectively. Then*

$$T\colon L_p(U, w d\mu)\to L_q(V, \tilde{w} dv)$$

where

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}, \quad p\leqslant q$$

and

$$w=w_0^{p(1-\theta)/p_0}w_1^{p\theta/p_1},$$
$$\tilde{w}=\tilde{w}_0^{q(1-\theta)/q_0}\tilde{w}_1^{q\theta/q_1}. \quad \square$$

Using the complex method, we can drop the restriction $p\leqslant q$ and we get sharper inequalities for the operator norms. However, we have to exclude the case $0<p_0, p_1<1$.

5.5.3. Theorem. *Assume that* $1\leqslant p_0, p_1<\infty$. *Then we have, with equal norms,*

$$(L_{p_0}(w_0), L_{p_1}(w_1))_{[\theta]}=L_p(w) \quad (0<\theta<1),$$

where

$$w=w_0^{p(1-\theta)/p_0}w_1^{p\theta/p_1}, \quad 1/p=(1-\theta)/p_0+\theta/p_1.$$

Proof: For a given $f\in\mathscr{F}(L_{p_0}(w_0), L_{p_1}(w_1))$, we put

$$\tilde{f}(z, x)=w_0(x)^{(1-z)/p_0}w_1(x)^{z/p_1}f(z, x).$$

The mapping $f\to\tilde{f}$ is obviously an isometric isomorphism between $(L_{p_0}(w_0), L_{p_1}(w_1))$ and $\mathscr{F}(L_{p_0}, L_{p_1})$. Now the argument in the proof of Theorem 5.1.1 goes through with evident modifications. $\square$

5.5.4. Corollary (Stein-Weiss). *Assume that* $1\leqslant p_0, p_1, q_0, q_1<\infty$, *and that*

$$T\colon L_{p_0}(U, w_0 d\mu)\to L_{q_0}(V, \tilde{w}_0 dv),$$
$$T\colon L_{p_1}(U, w_1 d\mu)\to L_{q_1}(V, \tilde{w}_1 dv),$$

with norms M_0 *and* M_1 *respectively. Then*

$$T\colon L_p(U, w d\mu)\to L_q(V, \tilde{w} dv)$$

with norm M, satisfying

$$M \leqslant M_0^{1-\theta} M_1^{\theta},$$

where

$$1/p=(1-\theta)/p_0+\theta/p_1, \qquad 1/q=(1-\theta)/q_0+\theta/q_1,$$
$$w=w_0^{p(1-\theta)/p_0} w_1^{p\theta/p_1}, \qquad \tilde{w}=\tilde{w}_0^{q(1-\theta)/q_0} \tilde{w}_1^{q\theta/q_1}. \quad \square$$

5.6. Interpolation of L_p-Spaces of Vector-Valued Sequences

It is possible to extend many of the previous results to vector valued L_p-spaces. (Cf. Theorem 5.1.2.) However, these extensions will be complicated by questions of measurability. In order to avoid these difficulties, we shall consider L_p-spaces of sequences only.

Let A be a Banach space and let s be an arbitrary real number and q a positive real number. Then we denote by $\dot{l}_q^s(A)$ the space of all sequences $(\alpha_v)_{-\infty}^{\infty}$, $a_v \in A$ such that

$$\|(a_v)\|_{\dot{l}_q^s(A)} = (\sum_{-\infty}^{\infty} (2^{vs} \|a_v\|_A)^q)^{1/q}$$

is finite. Clearly, $\dot{l}_q^s(A)$ is a quasi-normed space. Note that if $A=\mathbb{R}$ (= space of real numbers), then $\dot{l}_q^{-\theta}(\mathbb{R})=\lambda^{\theta,q}$. We also introduce the space $l_q^s(A)$ of all sequences $(a_v)_0^{\infty}$, $a_v \in A$ such that

$$\|(a_v)\|_{l_q^s(A)} = (\sum_{v=0}^{\infty} (2^{vs} \|a_v\|_A)^q)^{1/q} < \infty .$$

We shall also work with the space $\dot{c}_0^s(A)$ of all $(a_v)_{-\infty}^{\infty}$ such that $2^{vs}\|a_v\|_A \to 0$ as $v \to \pm\infty$ and the space $c_0^s(A)$ of all $(a_v)_0^{\infty}$ such that $2^{vs}\|a_v\|_A \to 0$ as $v \to \infty$. The norms on $\dot{c}_0^s(A)$ and $c_0^s(A)$ are the norms of $\dot{l}_{\infty}^s(A)$ and $l_{\infty}^s(A)$ respectively.

Let $\mathbb{N}$ denote the set of non-negative integers and $\mathbb{Z}$ the set of all integers. Let $d\mu$ be the measure $\sum_{v \geqslant 0} 2^{vs}\delta_v$ (δ_v = pointmass 1 at $x=v$) and $d\dot{\mu}$ the measure $\sum_v 2^{vs}\delta_v$. Then

$$l_q^{s/q}(A)=L_q(\mathbb{N}, d\mu; A), \qquad c_0^s(A)=L_{\infty}^0(\mathbb{N}, d\mu; A),$$
$$\dot{l}_q^{s/q}(A)=L_q(\mathbb{Z}, d\dot{\mu}; A), \qquad \dot{c}_0^s(A)=L_{\infty}^0(\mathbb{Z}, d\mu; A).$$

Using Theorem 5.1.2, we therefore obtain

$$(l_{q_0}^{s/q_0}(A_0), l_{q_1}^{s/q_1}(A_1))_{[\theta]} = l_q^{s/q}((A_0, A_1)_{[\theta]}),$$
$$(l_{q_0}^{s/q_0}(A_0), c_0^s(A_1))_{[\theta]} = l_q^{s/q}((A_0, A_1)_{[\theta]}),$$

and similarly for the dotted spaces $\dot{l}_q^s$ and $\dot{c}_0^s$. Here $1 \leqslant q_0 < \infty$, $1 \leqslant q_1 < \infty$ and $1/q=(1-\theta)/q_0+\theta/q_1$.

We shall now prove analogous results for the real interpolation method. We start with the case $A_0=A_1=A$.

5.6.1. Theorem. *Assume that* $0<q_0\leqslant\infty$, $0<q_1\leqslant\infty$, *and that* $s_0\neq s_1$. *Then we have, for all* $q\leqslant\infty$,

$$(1)\qquad (l^{s_0}_{q_0}(A),\, l^{s_1}_{q_1}(A))_{\theta,q}=l^s_q(A),$$

where

$$s=(1-\theta)s_0+\theta s_1\,.$$

If $s_0=s_1=s$ *we have*

$$(2)\qquad (l^s_{q_0}(A),\, l^s_{q_1}(A))_{\theta,q}=l^s_q(A)$$

provided that

$$\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}\,.$$

The same statements hold for the dotted spaces $\dot{l}^s_q$. *In the case* $q_0<q_1=\infty$, *we can replace* l^s_∞ *or* $\dot{l}^s_\infty$ *by* c^s_0 *and* $\dot{c}^s_0$ *respectively.*

Proof: We first consider the case $s_0\neq s_1$, $q_0=q_1=r<q$. Let a denote the sequence (a_ν). Then

$$\begin{aligned}K_r(t,a;\,l^{s_0}_r(A),\,l^{s_1}_r(A))&=(\textstyle\sum_\nu \inf_{a_\nu=a_{\nu 0}+a_{\nu 1}}(2^{\nu s_0}\|a_{\nu 0}\|_A)^r+(t2^{\nu s_1}\|a_{\nu 1}\|_A)^r)^{1/r}\\ &\sim(\textstyle\sum_\nu(\min(2^{\nu s_0},\, t2^{\nu s_1})\|a_\nu\|_A)^r)^{1/r}\,.\end{aligned}$$

Thus it follows that

$$\min(2^{\mu s_0},\, t2^{\mu s_1})\|a_\mu\|_A\leqslant CK_r(t,a;\,l^{s_0}_r(A),\,l^{s_1}_r(A))\,.$$

With $t=2^{\mu\lambda}$, $\lambda=s_0-s_1$, we now obtain

$$\|a_\mu\|_A\leqslant C2^{-\mu s_0}K_r(2^{\mu\lambda},a;\,l^{s_0}_r(A),\,l^{s_1}_r(A))\,,$$

and hence

$$(\textstyle\sum_\mu(2^{\mu s}\|a_\mu\|_A)^q)^{1/q}\leqslant C(\sum_\mu(2^{\mu(s-s_0)}K_r(2^{\mu\lambda},a;\,l^{s_0}_r(A),\,l^{s_1}_r(A)))^q)^{1/q}\,.$$

Just as in the proof of the reiteration theorem, we now obtain

$$(3)\qquad \|a\|_{l^s_q(A)}\leqslant C\|a\|_{(l^{s_0}_r(A),\,l^{s_1}_r(A))_{\theta,q}}\,,$$

provided that $s=(1-\theta)s_0+\theta s_1$. In order to prove the converse inequality, we note that if $\eta=s-s_0$ then

$$\|a\|_{(l_r^{s_0}(A),\, l_r^{s_1}(A))_{\theta,q}} \leqslant C\|K_r(2^{\mu\lambda}, a;\, l_r^{s_0}(A), l_r^{s_1}(A))\|_{l_q^\eta}$$
$$\leqslant C\|(\textstyle\sum_\nu(\min(1,2^{-\nu\lambda})2^{(\nu+\mu)s_0}\|a_{\nu+\mu}\|_A)^r)^{1/r}\|_{l_q^\eta}.$$

With $p=q/r>1$, we obtain, by Minkowski's inequality,

$$\|a\|_{(l_r^{s_0}(A),\, l_r^{s_1}(A))_{\theta,q}} \leqslant C(\textstyle\sum_\nu(\min(1,2^{-\nu\lambda})2^{-\nu\eta})^r)^{1/r}\|a\|_{l_q^s(A)}.$$

This gives the converse of (3).

In order to prove (1), we now use the reiteration theorem, writing $l_{q_0}^{s_0}=(l_r^{\sigma_0}, l_r^{\sigma_1})_{\theta_0,q_0}$ and $l_{q_1}^{s_1}=(l_r^{\sigma_0}, l_r^{\sigma_1})_{\theta_1,q_1}$, with $\sigma_0<\min(s_0,s_1)<\max(s_0,s_1)<\sigma_1$ and $s_j=(1-\theta_j)\sigma_0+\theta_j\sigma_1$. Since then $\theta_0\neq\theta_1$, we can apply the reiteration theorem, which clearly gives (1).

Using Theorem 5.2.4, we also get (2), since then we have to take $\theta_0=\theta_1$.

The last part of the theorem, concerning the spaces $c_0^s(A)$, follows from Theorem 3.4.2. It is clear that the proof also works for the dotted spaces. □

Using the idea from the first half of the proof of the previous theorem, we prove the following result.

5.6.2. Theorem. *Assume that* $0<q_0<\infty$ *and* $0<q_1<\infty$. *Then*

$$(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))_{\theta,q}=l_q^s((A_0, A_1)_{\theta,q}),$$

where $s=(1-\theta)s_0+\theta s_1$, $1/q=(1-\theta)/q_0+\theta/q_1$.

Proof: Clearly,

$$L(t,a)=K(t,a;(l_{q_0}^{s_0}(A_0))^{q_0}, (l_{q_1}^{s_1}(A_1))^{q_1})$$
$$=\textstyle\sum_\nu \inf_{a_\nu=a_{\nu 0}+a_{\nu 1}}((2^{\nu s_0}\|a_{\nu 0}\|_{A_0})^{q_0}+t(2^{\nu s_1}\|a_{\nu 1}\|_{A_1})^{q_1}),$$

and thus

$$\Phi_{\eta,1}(L(t,a))=\textstyle\sum_\nu 2^{\nu s_0 q_0}\int_0^\infty t^{-\eta}K(t2^{\nu(s_1q_1-s_0q_0)}, a_\nu;(A_0)^{q_0},(A_1)^{q_1})\,dt/t$$
$$=\textstyle\sum_\nu 2^{\nu sq}\|a_\nu\|_{((A_0)^{q_0},(A_1)^{q_1})_{\eta,1}}.$$

By the power theorem the result now follows. □

Finally, we give a result for the complex method.

5.6.3. Theorem. *We have, with equal norms,*

$$(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))_{[\theta]}=l_q^s(\bar{A}_{[\theta]})$$
$$(0<\theta<1;\ s_0, s_1\in\mathbb{R};\ 1\leqslant q_0, q_1\leqslant\infty),$$

where

$$1/q=(1-\theta)/q_0+\theta/q_1\,,$$
$$s=(1-\theta)s_0+\theta s_1\,.$$

Proof: Let $f\in\mathscr{F}(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))$, where $f(z)=(f_k(z))_{k=0}^{\infty}$. Then we define $\tilde{f}(z)=(\tilde{f}_k(z))_k$ by

$$\tilde{f}_k(z)=2^{s_0(1-z)k}2^{s_1 zk}f_k(z)\,,$$

imitating the proof of Theorem 5.5.3. The mapping $f\to\tilde{f}$ is an isometric isomorphism between $\mathscr{F}(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))$ and $\mathscr{F}(l_{q_0}(A_0), l_{q_1}(A_1))$. This implies the result in view of Theorem 5.1.2. □

5.7. Exercises

1. Consider a linear operator T, defined for complex-valued measurable functions on, e.g., the real axis, and with values in a Banach function space X.

A *Banach function space* is a Banach space of complex-valued measurable functions with the following properties:
(i) $|f(x)|\leqslant|g(x)|$ a.e., $g\in X\Rightarrow f\in X$, $\|f\|_X\leqslant\|g\|_X$;
(ii) $0\leqslant f_{n-1}\leqslant f_n$, $f_n\to f$ a.e., $f\in X\Rightarrow \sup_n\|f_n\|_X=\|f\|_X$.
Prove that $(0<p\leqslant1)$

$$T\colon L_{p1}\to X$$

iff

$$\|T\chi_E\|_X\leqslant C(\mu(E))^{1/p}\,,$$

where χ_E is the characteristic function of the measurable set E.

Hint: Consider first the simple functions.

2. (Holmstedt [1]). Prove that if the measure μ is non-atomic and $0<p_0<p_1<\infty$ then

$$K(t,f;L_{p_0},L_{p_1})\sim\left(\int_0^{t^\alpha}(f^*(s))^{p_0}ds\right)^{1/p_0}+\left(\int_{t^\alpha}^{\infty}(f^*(s))^{p_1}ds\right)^{1/p_1},$$

where $1/\alpha=1/p_0-1/p_1$, by straight-forward estimates.

Hint: Choose a set E such that $\mu(E)=t^\alpha$ and $|f(x)|\geqslant f^*(t^\alpha)$ for $x\in E$, and put

$$f_0(x)=\begin{cases}f(x) & x\in E\\ 0 & x\notin E\,.\end{cases}$$

3. (Holmstedt [1]). Prove that if $0<p_0<p_1<\infty$ then $(1/\alpha=1/p_0-1/p_1)$

$$K(t,f;L_{p_0q_0},L_{p_1q_1})\sim\left(\int_0^{t^\alpha}(s^{1/p_0}f^*(s))^{q_0}ds\right)^{1/q_0}+\left(\int_{t^\alpha}^{\infty}(s^{1/p_1}f^*(s))^{q_1}ds\right)^{1/q_1}.$$

In particular this gives a formula for $(q_i=p_i)$ $K(t,f;L_{p_0},L_{p_1})$.

Hint: Use 3.6.1, 5.2.1 and Minkowski's inequality.

4. (O'Neil [1]). Consider the convolution operator

$$Tf(x)=\int_{\mathbb{R}^n}k(x-y)f(y)\,dy.$$

Assume that $k\in L_\rho^*=L_{\rho\infty}$. Then

$$T\colon L_p\to L_q,$$

where $1/q=1/p-1/\rho'$ and $1<p<\rho'$.

Hint: Cf. the proof of Young's inequality 1.2.2, and use Theorem 5.3.1.

5. Define the potential operator P_α by

$$P_\alpha f(x)=\int_{\mathbb{R}^n}f(x-y)|y|^{\alpha-n}dy \qquad (0<\alpha<n).$$

Show that

$$P_\alpha\colon L_p\to L_q$$

if $1/q=1/p-\alpha/n$ and $1<p<n/\alpha$.

Hint: Use the previous exercise. (Cf. Peetre [29] for a detailed account.)

6. (Goulaouic [1]). Consider the positive real axis and the usual Lebesgue measure. Put $(0<\beta\leqslant\alpha)$

$$\begin{cases} w_0(x)=1\\ w_1(x)=\exp(x^\alpha)\end{cases}$$

and

$$h(n)=\begin{cases}\exp((\log n)^{\beta/\alpha}) & \text{if } n\geqslant 1\\ 0 & \text{if } n<1.\end{cases}$$

Show that if

$$T\colon L_p(w_i)\to L_q(w_i)\qquad (i=0,1)$$

then

$$T\colon L_p(w)\to L_p(w),$$

where

$$w(x)=\exp(x^\beta).$$

Hint: Show that h is concave and apply Theorem 5.5.4.

7. Consider the positive real axis and the usual Lebesgue measure. Put $A_0=L_1$, $A_1=L_\infty$, $X_i=\bar{A}_{\theta_i,q_i}$ $(i=0,1;\ 0<\theta_0<\theta_1<1)$. Show that if

$$f(x)=\begin{cases} x & \text{if } 0<x\leqslant 1 \\ 0 & \text{if } \quad x>1\end{cases}$$

then

$$\|f\|_{\bar{X}_{\eta,q}}\|f\|^{-1}_{\bar{A}_{\theta,q}}\sim(\eta(1-\eta))^{-1/q},$$

where $\theta=(1-\eta)\theta_0+\eta\theta_1$, $0<\eta<1$, $0<q\leqslant\infty$. Moreover, if

$$f(x)=\begin{cases} \theta_1 x^{\theta_1-1} & \text{if } 0<x\leqslant 1 \\ \theta_0 x^{\theta_0-1} & \text{if } \quad x>1\end{cases}$$

show that

$$\|f\|_{\bar{X}_{\eta,q}}\|f\|^{-1}_{\bar{A}_{\theta,q}}\sim\eta^{-1/q_0}(1-\eta)^{-1/q_1}.$$

Hint: Employ Theorem 3.6.1, 5.2.1 and Exercise 3 to estimate $K(t,f)$.

8. As an application of Exercise 16, Chapter 3, we suggest the following: Let Q be the semi-group of all non-negative decreasing sequences and consider Q as a sub-semi-group of l_∞.

(a) Prove that $(l_1\cap Q,\ l_\infty\cap Q)_{\theta,q}=l_{pq}\cap Q$ if $1/p=1-\theta$.

(b) Consider the operator T given by

$$(T(a_n)_1^\infty)(x)=\sup_{x\leqslant t\leqslant\pi}\left|\textstyle\sum_{n=1}^\infty a_n\cos nt\right| \qquad (x\geqslant 0).$$

Prove that

$$T\colon l_1\to L_\infty(0,\pi),$$
$$T\colon l_\infty\cap Q\to L_{1\infty}(0,\pi).$$

(c) Deduce that

$$(a_n)_1^\infty\in l_{pq}\cap Q\Rightarrow\textstyle\sum_{n=1}^\infty a_n\cos nx\in L_{p'q}(0,\pi),$$

$1/p'=1-1/p$, $1<p<\infty$. (This is a classical theorem by Hardy and Littlewood. For more general results of this kind see Y. Sagher [2, 3].)

9. Prove that $(L_{pq})'=L_{p'q'}$ if $1<p<\infty$, $1\leqslant q<\infty$, and (using Exercise 17, Chapter 3) that $(L_{pq})'=L_{p'\infty}$ if $1<p<\infty$, $0<q<1$. (Cf. Haaker [1] and Peetre [26] for $0<p<1$. See also Sagher [1] and Cwikel [2].)

10. Put

$$A(s,f)=\int_{sw_0(x)\leqslant w_1(x)}|f(x)|^p w_0(x)d\mu, \qquad 0<p<\infty,$$

and $K(t,f)=K(t,f;L_p(w_0),L_p(w_1))$. Prove that $(L_p(w_0),L_p(w_1))_{\theta,q}$ is the space of all measurable functions such that

$$(\int_0^\infty (s^\theta A(s,f))^{q/p}ds/s)^{1/q}<\infty.$$

11. Prove that $(L_p(w_0),L_p(w_1))_{\theta,q}$ is a retract of $l_q(L_p(w_\theta))$, where $w_\theta=w_0^{1-\theta}w_1^\theta$. (For the definition of retract see Exercise 18, Chapter 3.)

Hint: Exercise 10.

12. (Peetre [20]). Consider the couple (C^0,C^1) as defined in 7.6. Show that the mapping T defined by

$$Tf(x)=f(x)-f(y)$$

is an isometric (K-invariant) isomorphism between the couple (C^0,C^1) and a subcouple of the couple $(l_\infty(\frac{1}{2}),l_\infty(|x-y|^{-1}))$. Cf. 3.13.13.

Hint: Apply the formula for the K-functional and Exercise 3.13.1.

13. Assume that h is a non-negative function defined on the positive real axis. Show that the function k, defined by

$$k(t)=\sup\{\textstyle\sum_{i=1}^n\lambda_i h(t_i)\,|\,\sum\lambda_i=1,\ \sum\lambda_i t_i=t,\ \lambda_i\geqslant 0\}$$

is the least concave majorant of h.

14. (Sedaev-Semenov [1]). Consider the couple (A_0,A_1), where $A_0=A_1=\mathbb{R}^3$ (as sets) with norms

$$\|x\|_{A_0}=x_1^*+x_2^*,$$
$$\|x\|_{A_1}=x_1^*.$$

Here $(x_\nu^*)_1^3$ is the decreasing rearrangement of $(|x_\nu|)_1^3$ and $x=(x_1,x_2,x_3)\in\mathbb{R}^3$. Show that

$$K(t,x)=\begin{cases} tx_1^* & (0<t\leqslant 1)\\ x_1^*+(t-1)x_2^* & (1<t\leqslant 2)\\ x_1^*+x_2^* & (t>2).\end{cases}$$

Put $x=(3,2,1)$ and $y=(3,2,2)$. Evidently, $K(t,x)=K(t,y)$ for all $t>0$. Let the norm of A be defined by

$$\|z\|_A=\inf\{\max(\|T\|_0,\|T\|_1)\,|\,z=Tx,\ T\in L(\bar{A})\}\,.$$

Verify that A is an exact interpolation space with respect to $\bar{A}$, but that

$$\|x\|_A=1<\|y\|_A\,.$$

Hint: Show that $y\neq Tx$ when $\max(\|T\|_0,\|T\|_1)\leqslant 1$, $T\in L(\bar{A})$.

15. State and prove multilinear interpolation theorems for bounded linear mappings from products of L_p-spaces to L_p-spaces, using the real and the complex method.

Hint: See the general theorems and the exercises in Chapter 3 and 4.

5.8. Notes and Comment

As we noted in Chapter 1, the study of interpolation of L_p-spaces, or, rather, of operators between L_p-spaces, previews retrospectively the theorems of Riesz and Marcinkiewicz. It is these latter theorems and some of their generalizations which are the theme of this chapter—now, of course, seen in the light of the complex and real methods.

Other methods have been introduced by Bennett [2]. His methods are adapted to couples of rearrangement invariant Banach function spaces, and are equivalent to the real method.

Interpolation of Orlicz spaces is the subject of Gustavsson-Peetre [1]. They consider the problem of putting necessary and sufficient conditions on φ in order that L^φ be an interpolation space with respect to the couple $(L^{\varphi_0}, L^{\varphi_1})$. The corresponding problem for Orlicz classes is essentially solved in Peetre [18]. Bennett [1] has shown that $(L\log^+ L, L_\infty)_{\theta,p}\subset L_p$ (strict inclusion) if $1/p=1-\theta$, $0<\theta<1$.

Throughout this chapter we have identified spaces obtained from a given L_p-couple by the complex and the real interpolation methods. There is, of course, also a converse problem: *Can "all" interpolation spaces with respect to a fixed couple of L_p-spaces be obtained by the complex/real interpolation methods?* This problem, for the couple (L_1, L_∞) (L_∞ being the closure of the simple functions), has been treated by Mitjagin [1], and, later, by Calderón [3]. They showed that "all" interpolation spaces with respect to (L_1, L_∞) are K-spaces in the following sense: Assume that A is a rearrangement invariant Banach function space, which is an exact interpolation space with respect to (L_1, L_∞). Then

$$K(t,f;L_1,L_\infty)\leqslant K(t,g;L_1,L_\infty)\quad\text{for all}\quad t>0,\qquad g\in A$$
$$\Rightarrow f\in A\quad\text{and}\quad \|f\|_A\leqslant\|g\|_A\,.$$

The corresponding problem for the couple (L_p, L_∞) has been explored by Cotlar (personal communication) and Lorentz and Shimogaki [1] (see also Bergh [1]). Recently, Sparr [2] has shown that all quasi-Banach function spaces which are interpolation spaces with respect to the couple $(L_{p_0}(w_0), L_{p_1}(w_1))$, where $0<p_0$, $p_1<\infty$ and $\|f\|^p_{L_p(w)}=\int_0^\infty |f(x)w(x)|^p dx$, are equivalent to a K-space (in the sense above). Sparr's results contain a result by Sedaev [1]: the case $1\leqslant p_0=p_1<\infty$. Sedaev-Semenov [1] exhibits an example of a couple and an interpolation space with respect to that couple which is not a K-space (see Exercise 14). In other words: given an arbitrary couple, the K-method does not necessarily exhaust all interpolation spaces with respect to that couple. Moreover, returning to the original problem, Lorentz and Shimogaki [1] have given necessary and sufficient conditions for a Lorentz spaces $\Lambda(\varphi)$ to be an (exact) interpolation space with respect to the Lorentz couple $(\Lambda(\varphi_0), \Lambda(\varphi_1))$, where $\|f\|_{\Lambda(\varphi)}=\int_0^1 f^*(t)\varphi(t)dt$, φ being positive and decreasing. In fact, their conditions say that $\Lambda(\varphi)$ can be obtained from $(\Lambda(\varphi_0), \Lambda(\varphi_1))$ by the K-method, see Bergh [1]. Also, we remark that Lorentz and Shimogaki admit even Lipschitz operators, not only linear operators. (Cf. also 3.14 and 2.6, and especially the result of Cwikel concerning the K-monotonicity of the couple $(L_{p_0q_0}, L_{p_1q_1})$.)

5.8.1. Section 5.1 is essentially taken over from Calderón [2]. However, Calderón studies interpolation of general Banach function lattices and then simultaneously covers e.g. L_p-, L_{pq}- and Orlicz spaces L^φ. This is done by introducing a space A:

$$A=\{f\,|\,\exists\lambda>0,\ 0<\theta<1,\ f_i\in A_i;\ \|f_i\|_{A_i}\leqslant 1,\ |f(x)|\leqslant\lambda|f_0(x)|^{1-\theta}|f_1(x)|^\theta\},$$

where $\bar{A}$ is a compatible couple of Banach function lattices over the same measure space. A is denoted by $A_0^{1-\theta}A_1^\theta$ and is also a Banach function lattice with norm $\|f\|_A=\inf\lambda$. Under supplementary assumptions on $\bar{A}$ and using vector-valued functions, Calderón shows that, e.g., $(A_0(B_0), A_1(B_1))_{[\theta]}\subset A_0^{1-\theta}A_1^\theta((B_0, B_1)_{[\theta]})$, where $A(B)$ denotes the Banach space of B-valued measurable functions $f(x)$, such that $\|f(x)\|_B\in A$ and $\|f\|_{A(B)}=\|(\|f(x)\|_B)\|_A$. The inclusion becomes norm equality if $f\in A$, $|f_n|\leqslant|f|$, $f_n\to 0$ a.e. implies $\|f_n\|_A\to 0$. Also, he proves an analogous result for C^θ, and, using this, he gives an example of a couple for which $\bar{A}_{[\theta]}\neq\bar{A}^{[\theta]}$. However, Šestakov [1] has found that if $B_0=B_1=\mathbb{C}$ then $\bar{A}_{[\theta]}$ coincides with the closure of $\Delta(\bar{A})$ in $A_0^{1-\theta}A_1^\theta$, and that $\bar{A}_{[\theta]}$ is a closed subspace of $\bar{A}^{[\theta]}$, on $\bar{A}_{[\theta]}$ the norms being equal.

5.8.2—3. Section 5.2 and 5.3 are originally due to Lions-Peetre [1] and Peetre [10]. Formula 5.2.(1) in the case $p=1$ was found by Peetre [10], and, in general, by Krée [1] (see Oklander [1] and Bergh [2]). Also, the idea of considering the functional L is Peetre's [18]. (In connection with Theorem 5.2.4, see also Berenstein-Kerzman [1].)

The Lorentz spaces L_{pq} were introduced by G.G. Lorentz [1, 2] in 1953. Calderón [3] indicated their full significance in interpolation theory. (See also Krein [1] ($q=1$), and 1.7.) Sharpley [1] considers "weak interpolation" of a generalization of the spaces L_{pq}.

Using Formula 5.2.(1) and his Theorem 3.6.1, Holmstedt [1] proved the formulas in 5.7.2 and 5.7.3. In that paper, he was also able to determine with more precision the constants in the norm-inequalities of the generalizations of the Riesz (5.2.3) and the Marcinkiewicz (5.3.2) theorems. His tool was the sharp form of the reiteration theorem (cf. 3.13.15), and his results were that in the Riesz theorem (5.2.3) the constant is independent of θ, and in the Marcinkiewicz theorem (5.3.2), the constant is $O(\theta^{\alpha_0}(1-\theta)^{\alpha_1})$, where $\alpha_i=\min(0,1/s-1/q)+\min(0,1/p-1/r_i)$. In a sense, these constants are the best possible (see Exercise 5.7.7).

Let us also point out that there is a close connection between Theorem 5.2.1 and Hardy's inequality. General forms of Hardy's inequality are found in, e.g., Andersson [1], Tomaselli [1]. (Cf. also the proof of Theorem 1.3.1.)

5.8.4. In Section 5.4 the original result (5.4.1) is due to Stein and Weiss [1].

The interpolation functions have been treated in several papers, e.g., Foiaş and Lions [1], Donoghue [1], Peetre [13] (for more references see, in particular, Peetre [13]). Theorem 5.4.4 was given by Peetre [13]. For exact interpolation functions, i.e. $C=1$ in the norm-inequality, the problem of determining all interpolation functions is still open, except for $p=2$ (see Donoghue [1]).

Gilbert [1], following Peetre [6], has obtained a characterization in the off-diagonal case. Gilbert studied thus the spaces $(L_p(w_0), L_p(w_1))_{\theta,q}$ for $p\neq q$. In particular, he was able to complete the identification of the Beurling spaces with interpolation spaces $(L_p(w_0), L_p(w_1))_{\theta,q}$ begun by Peetre. Lemma 5.4.3 is in part due to Lorentz [3] (see also Peetre [16]). Note that the function k, constructed in the last part of the proof, is the least concave majorant of the function h; this is Exercise 12.

5.8.5. Theorem 5.5.1, varying both the measures and the exponents, was shown by Peetre [18].

Lizorkin [1] has characterized the interpolation space $(L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,q}$ as a certain weighted Lorentz space. Theorem 5.5.1 is thus a corollary of Lizorkin's result. However, Lizorkin considers only the case $1\leqslant p_0, p_1, q\leqslant\infty$.

5.8.6. Early results on interpolation of spaces of vector-valued functions are found in Gagliardo [1] and Lions-Peetre [1].

Note that Theorem 5.6.2 is valid also for non-discrete measures. For instance, we have

$$(L_{p_0}(A_0), L_{p_1}(A_1))_{\theta,p}=L_p((A_0, A_1)_{\theta,p}),$$

if $1/p=(1-\theta)/p_0+\theta p_1$ (cf. Lions-Peetre [1]). Cwikel [1] showed that there is no reasonable generalization of this formula to other values of p.

Chapter 6

Interpolation of Sobolev and Besov Spaces

We present definitions, interpolation results and various inclusion and trace theorems for the Sobolev and Besov spaces; our approach follows Peetre [5]. In the first section, we introduce briefly the Fourier multipliers on L_p, and we prove the Mihlin multiplier theorem. In Section 8, we discuss interpolation of semi-groups of operators. Many other topics are touched upon in the notes and comment, e.g., interpolation of Hardy spaces H_p.

6.1. Fourier Multipliers

Throughout the chapter we shall discourse within the framework of the tempered distributions. The test functions for the tempered distributions are the infinitely differentiable functions f from $\mathbb{R}^n$ to $\mathbb{C}$, such that

$$P_{m,\alpha}(f)=\sup_{x\in\mathbb{R}^n}(1+|x|)^m|D^\alpha f(x)|$$

is finite for all m and all α. By D^α we mean

$$D^\alpha f=\frac{1}{i^{|\alpha|}}\cdot\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$$

where $|\alpha|=\alpha_1+\cdots+\alpha_n$ is the order of the multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$, i.e. the order of the derivative $D^\alpha f$. The class of these functions f is denoted by $\mathscr{S}$; it is a topological vector space, the topology being given by the seminorms $P_{m,\alpha}(f)$, $m=0,1,\dots$, $|\alpha|\geqslant 0$. The dual of $\mathscr{S}$, the space of tempered distributions, is denoted by $\mathscr{S}'$.

The Fourier transform is defined on $\mathscr{S}$ by the formula

$$\mathscr{F}f(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^n}\exp(-i\langle x,\xi\rangle)f(x)dx.$$

By Fourier's inversion formula, we have

$$f(x)=\mathscr{F}^{-1}(\mathscr{F}f)(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}\exp(i\langle x,\xi\rangle)\hat{f}(\xi)d\xi.$$

On $\mathscr{S}'$, the Fourier transform is given by

$$\langle \mathscr{F} f, g\rangle = \langle f, \mathscr{F} g\rangle,$$

where $f\in\mathscr{S}'$ and $g\in\mathscr{S}$. This has a meaning, since $g\in\mathscr{S}$ implies that $\mathscr{F}g\in\mathscr{S}$. For a proof of this statement and for more facts concerning the tempered distributions, we refer to L. Schwartz [1].

In this chapter L_p will always mean $L_p(\mathbb{R}^n, dx)$, with norm $\|\cdot\|_p$: $1\leqslant p\leqslant\infty$. (For the case $0<p<1$, see Notes and Comment.) Moreover, derivatives are to be interpreted in the distribution sense: $D^\alpha f$ $(f\in\mathscr{S}')$ is given by

$$\langle D^\alpha f, g\rangle = (-1)^{|\alpha|}\langle f, D^\alpha g\rangle \qquad (g\in\mathscr{S}),$$

where, of course, $D^\alpha g$ are derivatives as above. Also, equality is to be interpreted in the distribution sense, i. e. $f_1=f_2$ means that $\langle f_1, g\rangle = \langle f_2, g\rangle$ for all $g\in\mathscr{S}$.

6.1.1. Definition. *Let $\rho\in\mathscr{S}'$. ρ is called a Fourier multiplier on L_p if the convolution $(\mathscr{F}^{-1}\rho)*f\in L_p$ for all $f\in\mathscr{S}$, and if*

$$\sup_{\|f\|_p=1}\|(\mathscr{F}^{-1}\rho)*f\|_p$$

is finite. The linear space of all such ρ is denoted by M_p; the norm on M_p is the above supremum, written $\|\cdot\|_{M_p}$.

Since $\mathscr{S}$ is dense in L_p $(1\leqslant p<\infty)$, the mapping from $\mathscr{S}$ to L_p: $f\to(\mathscr{F}^{-1}\rho)*f$ can be extended to a mapping from L_p to L_p with the same norm. We write $(\mathscr{F}^{-1}\rho)*f$ also for the values of the extended mapping.

For $p=\infty$ (as well as for $p=2$) we can characterize M_p. Considering the map: $f\to(\mathscr{F}^{-1}\rho)*f$ for $f\in\mathscr{S}$, we note that it commutes with the translations. Therefore, $\rho\in M_\infty$ iff

$$|\mathscr{F}^{-1}\rho*f(0)|\leqslant C\|f\|_\infty \qquad (f\in\mathscr{S}).$$

But this inequality also means that $\mathscr{F}^{-1}\rho$ is a bounded measure on $\mathbb{R}^n$. Thus M_∞ is equal to the space of all Fourier transforms of bounded measures. Moreover, $\|\rho\|_{M_\infty}$ is equal to the total mass of $\mathscr{F}^{-1}\rho$. In view of the inequality above and the Hahn-Banach theorem, we may extend the mapping $f\to\mathscr{F}^{-1}\rho*f$ from $\mathscr{S}$ to L_∞ to a mapping from L_∞ to L_∞ without increasing its norm. The extended mapping we also write as $f\to\mathscr{F}^{-1}\rho*f$ $(f\in L_\infty)$.

6.1.2. Theorem. *If $1/p+1/p'=1$ we have $(1\leqslant p\leqslant\infty)$*

(1) $\qquad M_p=M_{p'}$ *(equal norms).*

Moreover,

(2)
$$M_1=\{\rho\in\mathscr{S}' \mid \mathscr{F}^{-1}\rho \text{ is a bounded measure}\}$$
$$\|\rho\|_{M_1} = \text{total mass of } \mathscr{F}^{-1}\rho$$

and

(3) $M_2 = L_\infty$ *(equal norms).*

For the norms $(1 \leqslant p_0, p_1 \leqslant \infty)$

(4) $\|\rho\|_{M_p} \leqslant \|\rho\|_{M_{p_0}}^{1-\theta} \|\rho\|_{M_{p_1}}^{\theta} \quad (\rho \in M_{p_0} \cap M_{p_1})$

if $1/p = (1-\theta)/p_0 + \theta/p_1$ $(0 \leqslant \theta \leqslant 1)$. *In particular, the norm* $\|\cdot\|_p$ *decreases with p in the interval* $1 \leqslant p \leqslant 2$, *and*

(5) $M_1 \subset M_p \subset M_q \subset M_2 \quad (1 \leqslant p < q \leqslant 2)$.

Proof: Let $f \in L_p, g \in L_{p'}$ and $\rho \in M_p$. Hölder's inequality gives

$$|(\mathscr{F}^{-1}\rho) * f * g(0)| \leqslant \|(\mathscr{F}^{-1}\rho) * f\|_p \|g\|_{p'} \leqslant \|\rho\|_{M_p} \|f\|_p \|g\|_{p'}.$$

From this we infer that $\mathscr{F}^{-1}\rho * g \in L_{p'}$, and that $\rho \in M_{p'}$ with $\|\rho\|_{M_{p'}} \leqslant \|\rho\|_{M_p}$. This proves (1). The assertion (2) has already been established. Parseval's formula immediately gives (3). In fact,

$$\sup_{\|f\|_2 = 1} \|(\mathscr{F}^{-1}\rho) * f\|_2 = \sup_{\hat{f} \in L_2} \|\rho \hat{f}\|_2 / \|\hat{f}\|_2 = \|\rho\|_\infty.$$

Invoking the Riesz-Thorin theorem, (4) follows, since the mapping $f \to (\mathscr{F}^{-1}\rho) * f$ maps $L_{p_0} \to L_{p_0}$ with norm $\|\rho\|_{M_{p_0}}$ and $L_{p_1} \to L_{p_1}$ with norm $\|\rho\|_{M_{p_1}}$. Using (4) with $p_0 = p$, $p_1 = p'$, we see that

$$\|\rho\|_{M_q} \leqslant \|\rho\|_{M_p}, \quad p \leqslant q,$$

from which (5) follows. □

Considering (3) and (5), we may clearly multiply $\rho_1 \in M_p$ and $\rho_2 \in M_p$ $(1 \leqslant p \leqslant \infty)$ to get a new function $\rho \in L_\infty$: $\rho(\xi) = \rho_1(\xi) \cdot \rho_2(\xi)$. Obviously, we get $\rho \in M_p$ and

$$\|\rho\|_{M_p} \leqslant \|\rho_1\|_{M_p} \|\rho_2\|_{M_p}.$$

Note also that M_p is complete. Thus M_p is a Banach algebra under pointwise multiplication.

In order to clarify the next theorem we write $M_p = M_p(\mathbb{R}^n)$ for Fourier multipliers which are functions on $\mathbb{R}^n$. The theorem says that $M_p(\mathbb{R}^n)$ is isometrically invariant under affine transformations of $\mathbb{R}^n$.

6.1.3. Theorem. *Let* $a: \mathbb{R}^n \to \mathbb{R}^m$ *be a surjective affine transformation. Then the mapping* a^*, *defined by*

$$(a^* \rho)(\xi) = \rho(a(\xi)) \quad (\xi \in \mathbb{R}^n),$$

from $M_p(\mathbb{R}^m)$ *to* $M_p(\mathbb{R}^n)$ *is isometric. If* $m=n$, *the mapping* a^* *is bijective. In particular, we have*

$$\|\rho(t\cdot)\|_{M_p(\mathbb{R}^n)}=\|\rho(\cdot)\|_{M_p(\mathbb{R}^n)} \qquad (t\neq 0)$$

$$\|\rho(\langle x,\cdot\rangle)\|_{M_p(\mathbb{R}^n)}=\|\rho(\cdot)\|_{M_p(\mathbb{R})} \qquad (x\neq 0).$$

Proof: It is easy to see that M_p is isometrically invariant under nonsingular linear coordinate transformations. Therefore, we may choose coordinate systems in $\mathbb{R}^n$ and $\mathbb{R}^m$, such that $a(\xi)_j=\xi_j, j=1,\dots,m$. Then

$$a^*\rho=\rho\otimes 1,$$

where $\xi_1,\dots,\xi_m$ are acted on by ρ and the remaining $n-m\,(\geqslant 0)$ variables by 1. Thus, with $f\in\mathscr{S}$, we obtain

$$\|\mathscr{F}^{-1}(a^*\rho)*f\|_p=\|(\mathscr{F}^{-1}\rho\otimes\delta)*f\|_p\leqslant\|\rho\|_{M_p}\|f\|_p,$$

by inspection of the integrals. This gives

$$\|a^*\rho\|_{M_p(\mathbb{R}^n)}\leqslant\|\rho\|_{M_p(\mathbb{R}^m)}. \tag{6}$$

Finally, taking

$$f(x)=f_1(x_1,\dots,x_m)\,f_2(x_{m+1},\dots,x_n),$$

equality in (6) follows. □

The Fourier multipliers can be defined also on certain vector-valued L_p spaces. We will use results for Fourier multipliers on L_p with values in a Hilbert space. Therefore we consider only this case. Let H be a Hilbert space, and consider the space $\mathscr{S}(\mathbb{R}^n;H)=\mathscr{S}(H)$ of all mappings f from $\mathbb{R}^n$ to H, such that $(1+|x|)^m\|D^\alpha f(x)\|_H$ is bounded for each α and m. The space $L(\mathscr{S}(H_0),H_1)$ consists of all linear continuous mappings from $\mathscr{S}(H_0)$ to H_1, where H_0 and H_1 are Hilbert spaces. This space is $\mathscr{S}'$ if $H_0=H_1=\mathbb{C}$. Clearly, we may define the Fourier transform on $\mathscr{S}(H_0)$ and on $L(\mathscr{S}(H_0),H_1)$ in the same way as before. The integrals converge in H_0, and it is obvious that the inversion formula holds. We shall also use the notation $\mathscr{S}'(H_0,H_1)$ for $L(\mathscr{S}(H_0),H_1)$.

6.1.4. Definition. *Let* H_0 *and* H_1 *be two Hilbert spaces with norms* $\|\cdot\|_0$ *and* $\|\cdot\|_1$ *respectively. Consider a mapping* $\rho\in\mathscr{S}'(H_0,H_1)$. *We write* $\rho\in M_p(H_0,H_1)$ *if, for all* $f\in\mathscr{S}(H_0)$, *we have* $(\mathscr{F}^{-1}\rho)*f\in L_p(H_1)$ *and if the expression*

$$\sup_{\|f\|_{L_p(H_0)}=1}\|(\mathscr{F}^{-1}\rho)*f\|_{L_p(H_1)}$$

is finite. The last expression is the norm, $\|\rho\|_{M_p(H_0,H_1)}$, *in* $M_p(H_0,H_1)$.

Theorems 6.1.2 and 6.1.3 have obvious analogues in this general situation. The proofs are the same with trivial changes.

6.1.5. Lemma. *Assume that* L *is an integer,* $L>n/2$, *and that* $\rho\in L_2(L(H_0,H_1))$ *and* $D^\alpha\rho\in L_2(L(H_0,H_1))$, $|\alpha|=L$. *Then* $\rho\in M_p(H_0,H_1)$, $1\leqslant p\leqslant\infty$, *and*

$$\|\rho\|_{M_p}\leqslant C\|\rho\|_{L_2}^{1-\theta}(\sup_{|\alpha|=L}\|D^\alpha\rho\|_{L_2})^\theta,$$

where $\theta=n/2L$.

Proof: Clearly, $\rho\in\mathscr{S}'(H_0,H_1)$. Let $t>0$. By the Cauchy-Schwarz inequality and the Parseval formula, we obtain

$$\begin{aligned}\int_{|x|>t}\|\mathscr{F}^{-1}\rho(x)\|_{L(H_0,H_1)}dx&\leqslant\int_{|x|>t}|x|^{-L}|x|^L\|\mathscr{F}^{-1}\rho(x)\|_{L(H_0,H_1)}dx\\&\leqslant Ct^{n/2-L}\sup_{|\alpha|=L}\|D^\alpha\rho\|_{L_2(L(H_0,H_1))}.\end{aligned}$$

Similarly, we have

$$\int_{|x|<t}\|\mathscr{F}^{-1}\rho(x)\|_{L(H_0,H_1)}dx\leqslant Ct^{n/2}\|\rho\|_{L_2(L(H_0,H_1))}.$$

Choosing t such that $\|\rho\|_{L_2}=t^{-L}\sup_{|\alpha|=L}\|D^\alpha\rho\|_{L_2}$, we infer that

$$\|\rho\|_{M_p}\leqslant\|\rho\|_{M_1}=\int_{\mathbb{R}^n}\|\mathscr{F}^{-1}\rho(x)\|_{L(H_0,H_1)}dx\leqslant C\|\rho\|_{L_2}^{1-\theta}(\sup_{|\alpha|=L}\|D^\alpha\rho\|_{L_2})^\theta.$$

Here the first inequality is a consequence of Theorem 6.1.2. □

Our main tool when proving theorems for the Sobolev and Besov spaces is the following theorem. Note that $1<p<\infty$ here in contrast to the case in Lemma 6.1.5.

6.1.6. Theorem (The Mihlin multiplier theorem). *Let* H_0 *and* H_1 *be Hilbert spaces. Assume that* ρ *is a mapping from* $\mathbb{R}^n$ *to* $L(H_0,H_1)$ *and that*

$$|\xi|^{|\alpha|}\|D^\alpha\rho(\xi)\|_{L(H_0,H_1)}\leqslant A\qquad(|\alpha|\leqslant L)\tag{7}$$

for some integer $L>n/2$. *Then* $\rho\in M_p(H_0,H_1)$, $1<p<\infty$, *and*

$$\|\rho\|_{M_p}\leqslant C_pA.$$

In the proof we use the following two lemmas. The first is frequently used later, and the second is essential to the proof of the theorem.

6.1.7. Lemma. *There exists a function* $\varphi\in\mathscr{S}(\mathbb{R}^n)$, *such that*

$$\operatorname{supp}\varphi=\{\xi\,|\,2^{-1}\leqslant|\xi|\leqslant2\}\tag{8}$$

(9) $\varphi(\xi)>0$ *for* $2^{-1}<|\xi|<2$

(10) $\sum_{k=-\infty}^{\infty}\varphi(2^{-k}\xi)=1 \quad (\xi\neq 0).$

Proof: Choose any function $f\in\mathscr{S}$, such that (8) and (9) are satisfied. Then

$$\operatorname{supp} f(2^{-k}\xi)=\{\xi|2^{k-1}\leqslant|\xi|\leqslant 2^{k+1}\}.$$

Therefore the sum

$$F(\xi)=\sum_{k=-\infty}^{\infty}f(2^{-k}\xi)$$

contains at most two non-vanishing terms for each $\xi\neq 0$. Clearly, $F\in\mathscr{S}$, and $F(\xi)>0$ for $\xi\neq 0$. Put $\varphi=f/F$. Clearly, $\varphi\in\mathscr{S}$, and satisfies (8) and (9). Since $F(2^{-j}\xi)=F(\xi)$, φ also satisfies (10). □

6.1.8. Lemma. *Let $f\in L_1$ and $\sigma>0$. Then there are cubes I_ν, $\nu=1,2,\ldots$, with disjoint interiors and with edges parallel to the coordinate axes, such that*

$$\sigma<\mu(I_\nu)^{-1}\int_{I_\nu}|f(x)|dx\leqslant 2^n\sigma,$$

$$|f(x)|\leqslant\sigma \quad \text{a.e.} \quad x\notin\bigcup_{\nu=1}^{\infty}I_\nu.$$

Proof: Choose cubes $I_\nu^{(0)}$ $(\nu=1,2,\ldots)$ with disjoint interiors and edges parallel to the coordinate axes, and such that

(11) $\mu(I_\nu^{(0)})^{-1}\int_{I_\nu^{(0)}}|f(x)|\,dx\leqslant\sigma.$

Split each $I_\nu^{(0)}$ into 2^n congruent cubes. These we denote by $I_\nu^{(1)}$, $\nu=1,2,\ldots$. There are two possibilities: either

$$\mu(I_\nu^{(1)})^{-1}\int_{I_\nu^{(1)}}|f(x)|\,dx\leqslant\sigma$$

or

$$\mu(I_\nu^{(1)})^{-1}\int_{I_\nu^{(1)}}|f(x)|\,dx>\sigma.$$

In the first case we split $I_\nu^{(1)}$ again into 2^n congruent cubes to get $I_\nu^{(2)}$ $(\nu=1,2,\ldots)$. In the second case we have

$$\sigma<\mu(I_\nu^{(1)})^{-1}\int_{I_\nu^{(1)}}|f(x)|dx\leqslant 2^n\sigma$$

in view of (11), and then we take $I_\nu^{(1)}$ as one of the cubes I_ν. A repetition of this argument shows that if $x\notin\bigcup_{\nu=1}^{\infty}I_\nu$ then $x\in I_{\nu_j}^{(j)}$ $(j=0,1,2,\ldots)$ for which

$$\mu(I_{\nu_j}^{(j)})\to 0$$

and

$$\mu(I_{\nu_j}^{(j)})^{-1}\int_{I_{\nu_j}^{(j)}}|f(x)|\,dx\leqslant\sigma \qquad (j=0,1,\ldots).$$

Thus $|f(x)|\leqslant\sigma$ a.e. $x\notin\bigcup_{v=1}^{\infty}I_v$ by Lebesgue's differentiation theorem (see, e.g., Dunford and Schwartz [1]). □

Proof of Theorem 6.1.6: Obviously, $\rho\in\mathscr{S}'(H_0,H_1)$, and, taking $\alpha=0$ in (7), also (Theorem 6.1.2)

$$\mathscr{F}^{-1}\rho*: L_2(H_0)\to L_2(H_1).$$

If, in addition, we prove

$$\mathscr{F}^{-1}\rho*: L_1(H_0)\to L_{1\infty}(H_1), \tag{12}$$

then it follows that

$$\mathscr{F}^{-1}\rho*: L_p(H_0)\to L_p(H_1) \qquad (1<p<2)$$

by the Marcinkiewicz theorem, and thus $\rho\in M_p(H_0,H_1)$ $(1<p<\infty)$ by Theorem 6.1.2.

In order to simplify the notation, we shall give the rest of the proof for the case $H_0=H_1=\mathbb{C}$.

Thus, we need prove only (12). For $f\in\mathscr{S}$, (12) takes the form

$$\sigma m(\sigma,\mathscr{F}^{-1}\rho*f)\leqslant C\|f\|_1 \qquad (\sigma>0). \tag{13}$$

Now we decompose f into two terms ($\sigma>0$ fixed):

$$f_0(x)=\begin{cases} f(x)-\mu(I_v)^{-1}\int_{I_v}f(t)dt, & x\in I_v, \quad v=1,2,\dots,\\ 0 \quad \text{elsewhere}\end{cases}$$

$$f_1(x)=\begin{cases} \mu(I_v)^{-1}\int_{I_v}f(t)dt, & x\in I_v, \quad v=1,2,\dots,\\ f(x) \quad \text{elsewhere}\end{cases}$$

where I_v are the cubes of Lemma 6.1.8. Since

$$m(\sigma,\mathscr{F}^{-1}\rho*f)\leqslant m(\sigma/2,\mathscr{F}^{-1}\rho*f_0)+m(\sigma/2,\mathscr{F}^{-1}\rho*f_1)$$

for any decomposition $f=f_0+f_1$, it is enough to prove (13) with the functions f_0 and f_1 respectively on the left hand side.

In order to estimate $m(\sigma/2,\mathscr{F}^{-1}\rho*f_0)$, we first note that the mean value of f_0 over each I_v vanishes. We have, with a_v as centres in I_v and $2I_v$ as the result of enlarging I_v to double its edges,

$$\begin{aligned} &m(\sigma/2,\mathscr{F}^{-1}\rho*f_0)\\ &\quad\leqslant\sigma(\{|\mathscr{F}^{-1}\rho*f_0|\geqslant\sigma/2\}\cap(\mathbb{R}^n\setminus\textstyle\bigcup_{v=1}^{\infty}2I_v))+\mu(\bigcup_{v=1}^{\infty}2I_v).\end{aligned} \tag{14}$$

Now, since the mean value of f_0 over I_ν vanishes,

$$\begin{aligned}(15)\qquad &\int_{\mathbb{R}^n\setminus\bigcup_{\nu=1}^{\infty}2I_\nu}|\mathscr{F}^{-1}\rho*f_0(x)|\,dx\\ &\leqslant\sum_{\nu=1}^{\infty}\int_{I_\nu}\Big(\int_{\mathbb{R}^n\setminus\bigcup_{\nu=1}^{\infty}2I_\nu}|\mathscr{F}^{-1}\rho(x-y)-\mathscr{F}^{-1}\rho(x-a_\nu)|\,dx\Big)|f_0(y)|\,dy\\ &\leqslant C\sum_{\nu=1}^{\infty}\int_{I_\nu}|f_0(y)|\,dy\leqslant C\,\|f\|_1,\end{aligned}$$

if we prove that

$$(16)\qquad \int_{\mathbb{R}^n\setminus\bigcup_{\nu=1}^{\infty}2I_\nu}|\mathscr{F}^{-1}\rho(x-y)-\mathscr{F}^{-1}\rho(x-a_\nu)|\,dx\leqslant C,\qquad y\in I_\nu.$$

We postpone the proof of (16) in order to conclude the estimates. By (14), (15), (16) and Lemma 6.1.8, we obtain

$$(17)\qquad m(\sigma/2,\mathscr{F}^{-1}\rho*f_0)\leqslant C\sigma^{-1}\|f\|_1+2^n\sum_{\nu=1}^{\infty}\mu(I_\nu)\leqslant C\sigma^{-1}\|f\|_1.$$

By the inclusion $L_2\subset L_{2\infty}$, $\varphi\in M_2$ and Lemma 6.1.8, it follows that

$$\begin{aligned}(18)\qquad \sigma^2 m(\sigma/2,\mathscr{F}^{-1}\rho*f_1)&\leqslant C\,\|\mathscr{F}^{-1}\rho*f_1\|_2^2=C\,\|f_1\|_2^2\\ &=C\Big\{\sum_{\nu=1}^{\infty}\mu(I_\nu)^{-1}\Big|\int_{I_\nu}f(x)\,dx\Big|^2+\int_{\mathbb{R}^n\setminus\bigcup_{\nu=1}^{\infty}I_\nu}|f(x)|^2\,dx\Big\}\\ &\leqslant C\sigma\Big\{\sum_{\nu=1}^{\infty}\Big|\int_{I_\nu}f(x)\,dx\Big|+\int_{\mathbb{R}^n\setminus\bigcup_{\nu=1}^{\infty}I_\nu}|f(x)|\,dx\Big\}\\ &\leqslant C\sigma\,\|f\|_1.\end{aligned}$$

The estimates (17) and (18) yield (13) as we noted earlier.

There remains the proof of (16). Clearly, it is sufficient to prove that

$$(19)\qquad \int_{|x|\geqslant 2t}|\mathscr{F}^{-1}\rho(x-y)-\mathscr{F}^{-1}\rho(x)|\,dx\leqslant CA\qquad (|y|\leqslant t,\ t>0).$$

We may obviously assume that $\rho(0)=0$. Then, writing $\rho_k(\xi)=\varphi(2^{-k}\xi)\rho(\xi)$, we have $\sum_{k=-\infty}^{\infty}\rho_k=\rho$. Thus

$$\begin{aligned}&\int_{|x|\geqslant 2t}|\mathscr{F}^{-1}\rho(x-y)-\mathscr{F}^{-1}\rho(x)|\,dx\\ &\qquad\leqslant\sum_{-\infty}^{\infty}\int_{|x|\geqslant 2t}|\mathscr{F}^{-1}\rho_k(x-y)-\mathscr{F}^{-1}\rho_k(x)|\,dx,\end{aligned}$$

and (19) will follow if we prove

$$\begin{aligned}(20)\qquad &\int_{|x|\geqslant 2t}|\mathscr{F}^{-1}\rho_k(x-y)-\mathscr{F}^{-1}\rho_k(x)|\,dx\\ &\leqslant CA\min((t\cdot 2^k)^{n/2-L},t\cdot 2^k)\qquad (|y|\leqslant t,\ L>n/2).\end{aligned}$$

To complete the proof, we now have to prove (20). Using the Cauchy-Schwarz inequality and Parseval's formula, we get

$$\begin{aligned}\int_{|x|\geq 2t}|\mathscr{F}^{-1}\rho_k(x-y)-\mathscr{F}^{-1}\rho_k(x)|dx &\leqslant 2\int_{|x|\geq t}|\mathscr{F}^{-1}\rho_k(x)|dx\\ &\leqslant C(\int_{|x|\geq t}|2^kx|^{-2L}dx)^{1/2}(\int_{|x|\geq t}|2^kx|^{2L}|\mathscr{F}^{-1}\rho_k|^2dx)^{1/2}\\ &\leqslant C(t\cdot 2^k)^{n/2-L}2^{-kn/2}(\int_{2^{k-1}<|\xi|<2^{k+1}}\sum_{|\alpha|=L}2^{2k|\alpha|}|D^\alpha\rho_k(\xi)|^2d\xi)^{1/2}\\ &\leqslant CA(t\cdot 2^k)^{n/2-L},\end{aligned}$$

since, as is easily verified, $|\xi|^\alpha|D^\alpha\rho_k(\xi)|\leqslant CA$ ($|\alpha|\leqslant L, \xi\in\mathbb{R}^n, k=0,\pm 1,\ldots$). Similarly, we obtain ($|y|\leqslant t$)

$$\begin{aligned}&\int_{|x|\geq 2t}|\mathscr{F}^{-1}\rho_k(x-y)-\mathscr{F}^{-1}\rho_k(x)|dx\\ &\leqslant\int_0^1\int_{\mathbb{R}^n}|\langle y,\operatorname{grad}\mathscr{F}^{-1}\rho_k(x-\tau y)\rangle|dxd\tau\leqslant Ct\sum_{j=1}^n\|\partial\mathscr{F}^{-1}\rho_k/\partial x_j\|_1\\ &\leqslant Ct\sum_{j=1}^n(\int(1+|2^kx|^2)^{-L}dx)^{1/2}(\int(1+|2^kx|^2)^L|\partial\mathscr{F}^{-1}\rho_k/\partial x_j|^2dx)^{1/2}\\ &\leqslant Ct\sum_{j=1}^n 2^{-kn/2}(\int_{2^{k-1}<|\xi|<2^{k+1}}\sum_{|\alpha|\leqslant L}2^{2k|\alpha|}|D^\alpha\xi_j\rho_k(\xi)|^2d\xi)^{1/2}\\ &\leqslant CAt\cdot 2^k.\end{aligned}$$

The last two estimates give (20), and so the proof is complete. □

6.2. Definition of the Sobolev and Besov Spaces

We give Peetre's definition of the Besov spaces, and also a definition of the generalized Sobolev spaces.

Using the standard function φ of Lemma 6.1.7 we define functions φ_k and ψ by

$$\mathscr{F}\varphi_k(\xi)=\varphi(2^{-k}\xi)\qquad(k=0,\pm 1,\pm 2,\ldots)$$

$$\mathscr{F}\psi(\xi)=1-\sum_{k=1}^\infty\varphi(2^{-k}\xi).$$

Evidently, $\varphi_k\in\mathscr{S}$ and $\psi\in\mathscr{S}$.

Moreover, we shall use two operators J^s and I^s, both from $\mathscr{S}'$ to $\mathscr{S}'$, defined by

$$J^sf=\mathscr{F}^{-1}\{(1+|\cdot|^2)^{s/2}\mathscr{F}f\}\qquad(s\in\mathbb{R}, f\in\mathscr{S}')$$

$$I^sf=\mathscr{F}^{-1}\{|\cdot|^s\mathscr{F}f\}\qquad(s\in\mathbb{R}, f\in\mathscr{S}', 0\notin\operatorname{supp}\mathscr{F}f).$$

The operators J^{-s} and I^{-s} are often called the Bessel and Riesz potentials of order s respectively.

Some simple properties of the objects just defined are collected in the following lemma.

6.2.1. Lemma. *Let* $f \in \mathscr{S}'$, *and assume that* $\varphi_k * f \in L_p$. *Then* $(1 \leqslant p \leqslant \infty; s \in R)$

$$\|J^s \varphi_k * f\|_p \leqslant C 2^{sk} \|\varphi_k * f\|_p \quad (k \geqslant 1), \tag{1}$$

$$\|I^s \varphi_k * f\|_p \leqslant C 2^{sk} \|\varphi_k * f\|_p \quad (all\ k), \tag{2}$$

and in addition if $\psi * f \in L_p$

$$\|J^s \psi * f\|_p \leqslant C \|\psi * f\|_p \tag{3}$$

where the constants C are independent of p and k.

Proof: Note that

$$\varphi_k * f = \textstyle\sum_{l=-1}^{1} \varphi_{k+l} * \varphi_k * f$$

holds for all k. Thus, if we establish that

$$\|\mathscr{F}(J^s \varphi_{k+l})\|_{M_p} \leqslant C 2^{ks} \tag{4}$$

$$\|\mathscr{F}(I^s \varphi_{k+l})\|_{M_p} \leqslant C 2^{ks} \tag{5}$$

$(l = 0, \pm 1)$

then (1) and (2) follow.

To prove (4), we note that the function

$$\mathscr{F}\{J^s \varphi_{k+l}\} = (1 + |\cdot|^2)^{s/2} \mathscr{F} \varphi_{k+l} = (1 + |\cdot|^2)^{s/2} \varphi(2^{-(k+l)}\cdot)$$

has the same norm in M_p as the function

$$2^{(k+l)s}(2^{-2(k+l)} + |\cdot|^2)^{s/2} \varphi(\cdot)$$

by Lemma 6.1.2. Using Lemma 6.1.5, it is evident that the latter function in fact belongs to M_p with norm at most $C 2^{ks}$ $(k \geqslant 1)$, and thus (4) is established.

Formula (5) is easily proved in a similar way.

Finally, to prove (3), we note that

$$\psi * f = (\psi + \varphi_1) * \psi * f.$$

We need prove only that $\mathscr{F}(J^s \psi) \in M_p$, but this is obvious by Lemma 6.1.5. □

The previous lemma provides a background for the following definition of the Besov and the (generalized) Sobolev spaces.

6.2.2. Definition. *Let* $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. *We write*

$$\|f\|_{pq}^s = \|\psi * f\|_p + (\textstyle\sum_{k=1}^{\infty} (2^{sk} \|\varphi_k * f\|_p)^q)^{1/q},$$

$$\|f\|_p^s = \|J^s f\|_p.$$

The Besov space B^s_{pq} *and the generalized Sobolev space* H^s_p *are defined by*

$$B^s_{pq} = \{f : f \in \mathscr{S}', \|f\|^s_{pq} < \infty\},$$
$$H^s_p = \{f : f \in \mathscr{S}', \|f\|^s_p < \infty\}.$$

Clearly, B^s_{pq} and H^s_p are normed linear spaces with norms $\|\cdot\|^s_{pq}$ and $\|\cdot\|^s_p$ respectively. Moreover, they are complete and therefore Banach spaces. To prove that H^s_p is complete, let (f_n) be a Cauchy sequence in H^s_p. Then $g \in L_p$ exists (L_p is complete) such that

$$\|f_n - J^{-s}g\|^s_p = \|J^s f_n - g\|_p \to 0 \qquad (n \to \infty).$$

Clearly, $J^{-s}g \in \mathscr{S}'$ and thus H^s_p is complete. From this, Theorem 3.4.2 and (10) below, it follows that B^s_{pq} is complete. Thus H^s_p and B^s_{pq} are Banach spaces. Finally, the definition of B^s_{pq} does not depend on the choice of the function φ, i.e. with another choice we get an equivalent norm. This follows similarly from (10) below.

We give some elementary results about H^s_p and B^s_{pq}. First we consider H^s_p. In the following theorem there appears an alternative definition of H^s_p for positive integral values of s in terms of the derivatives $D^\alpha f$ ($|\alpha| \leqslant s$) of $f \in H^s_p$. (Note that $H^0_p = L_p$ $(1 \leqslant p \leqslant \infty)$.)

6.2.3. Theorem. *If* $s_1 < s_2$ *we have*

$$H^{s_2}_p \subset H^{s_1}_p \qquad (1 \leqslant p \leqslant \infty).$$

Moreover, if $N \geqslant 1$ *is an integer and if* $1 < p < \infty$ *then*

$$H^N_p = \{f \in L_p \mid \partial^N f / \partial x^N_j \in L_p, \ (1 \leqslant j \leqslant n)\}$$

and

$$\|f\|^N_p \sim \textstyle\sum_{j=1}^n \|\partial^N f / \partial x^N_j\|_p + \|f\|_p. \tag{6}$$

Finally, $\mathscr{S}$ *is dense in* H^s_p $(1 \leqslant p < \infty)$.

Proof: Suppose that $f \in H^{s_2}_p$. We shall see that $J^{s_1 - s_2}$ maps L_p into L_p. From this we get the first part of the theorem, since

$$\|f\|^{s_1}_p = \|J^{s_1} f\|_p = \|J^{s_1 - s_2} J^{s_2} f\|_p \leqslant C \|J^{s_2} f\|_p = C \|f\|^{s_2}_p.$$

In order to see that $J^{-\varepsilon}: L_p \to L_p$ if $\varepsilon = s_2 - s_1 > 0$, we use Lemma 6.2.1 and obtain

$$\|J^{-\varepsilon} f\|_p \leqslant \|J^{-\varepsilon}\psi * f\|_p + \textstyle\sum_{k=1}^\infty \|J^{-\varepsilon}\varphi_k * f\|_p$$
$$\leqslant C(\|\psi * f\|_p + \textstyle\sum_{k=1}^\infty 2^{-\varepsilon k} \|\varphi_k * f\|_p) \leqslant C(1 + \sum_{k=1}^\infty 2^{-\varepsilon k}) \|f\|_p.$$

This completes the proof of the first part of the theorem.

To prove the second part, we invoke the Mihlin multiplier theorem to obtain $\xi_j^N(1+|\xi|^2)^{-N/2}\in M_p\ (1<p<\infty)$. Thus

$$\begin{aligned}\|\partial^N f/\partial x_j^N\|_p &= \|\mathscr{F}^{-1}\{\xi_j^N \mathscr{F} f\}\|_p\\ &= \|\mathscr{F}^{-1}\{\xi_j^N(1+|\xi|^2)^{-N/2}\mathscr{F}(J^N f)\}\|_p \leq C\,\|f\|_p^N \quad (1\leq j\leq n).\end{aligned}$$

This gives one half of (6). For the other half of (6), we use Mihlin's multiplier theorem once more and an auxiliary function χ on $\mathbb{R}$, infinitely differentiable, non-negative and with $\chi(x)=1$ for $|x|>2$ and $\chi(x)=0$ for $|x|<1$. We obtain

$$\left.\begin{aligned}&(1+|\xi|^2)^{N/2}(1+\textstyle\sum_{j=1}^n \chi(\xi_j)|\xi_j|^N)^{-1}\in M_p\\ &\chi(\xi_j)|\xi_j|^N\xi_j^{-N}\in M_p\end{aligned}\right\}\quad (1<p<\infty).$$

Thus

$$\begin{aligned}\|J^N f\|_p &\leq C\,\|\mathscr{F}^{-1}\{(1+\textstyle\sum_{j=1}^N \chi(\xi_j)|\xi_j|^N)\mathscr{F} f\}\|_p\\ &\leq C(\|f\|_p+\textstyle\sum_{j=1}^n \|\mathscr{F}^{-1}\{\chi(\xi_j)|\xi_j|^N\xi_j^{-N}\mathscr{F}(\partial^N f/\partial x_j^N)\}\|_p)\\ &\leq C(\|f\|_p+\textstyle\sum_{j=1}^n \|\partial^N f/\partial x_j^N\|_p).\end{aligned}$$

It remains to prove the density. Take $f\in H_p^s$, i. e. $J^s f\in L_p$. Since $\mathscr{S}$ is dense in $L_p\ (1\leq p<\infty)$, there exists a $g\in\mathscr{S}$, such that

$$\|f-J^{-s}g\|_p^s=\|J^s f-g\|_p$$

is smaller than any given positive number. Since $J^{-s}g\in\mathscr{S}$, $\mathscr{S}$ is therefore dense in H_p^s. ☐

The results for the Besov spaces B_{pq}^s correspond in part to the previous theorem for the Sobolev spaces H_p^s. We split up these results into two theorems.

6.2.4. Theorem. *If* $s_1<s_2$ *we have*

(7) $$B_{pq}^{s_2}\subset B_{pq}^{s_1}\quad (1\leq p,q\leq\infty).$$

If $1\leq q_1<q_2\leq\infty$ *we have*

(8) $$B_{pq_1}^s\subset B_{pq_2}^s\quad (s\in\mathbb{R},\ 1\leq p\leq\infty).$$

Moreover,

(9) $$B_{p1}^s\subset H_p^s\subset B_{p\infty}^s\quad (s\in\mathbb{R},\ 1\leq p\leq\infty).$$

If $s_0\neq s_1$ *we also have*

(10) $$(H_p^{s_0},H_p^{s_1})_{\theta,q}=B_{pq}^s\quad (1\leq p,q\leq\infty,\ 0<\theta<1),$$

where $s=(1-\theta)s_0+\theta s_1$. *Finally, if* $1\leq p,q<\infty$ *then* $\mathscr{S}$ *is dense in* B_{pq}^s.

Proof: The formulas (7) and (8) follow at once from the definition of B^s_{pq}. The density statement is a consequence of (10), Theorem 3.4.2 and Theorem 6.2.3. The inclusions (9) are obviously implied by the inequalities in Lemma 6.2.1.

It remains to prove (10). Let $f \in (H^{s_0}_p, H^{s_1}_p)_{\theta,q}$, and put $f=f_0+f_1$, $f_i \in H^{s_i}_p$ $(i=0,1)$. By Lemma 6.2.1, we obtain

$$\|\varphi_k * f\|_p \leqslant \|\varphi_k * f_0\|_p + \|\varphi_k * f_1\|_p \leqslant C(2^{-s_0 k}\|J^{s_0} f_0\|_p + 2^{-s_1 k}\|J^{s_1} f_1\|_p),$$

and, taking the infinum,

$$\|\varphi_k * f\|_p \leqslant C 2^{-s_0 k} K(2^{k(s_0-s_1)}, f; H^{s_0}_p, H^{s_1}_p).$$

This gives

$$(\textstyle\sum_{k=1}^{\infty}(2^{sk}\|\varphi_k * f\|_p)^q)^{1/q} \leqslant C\|f\|_{(H^{s_0}_p, H^{s_1}_p)_{\theta,q}}.$$

Similarly, we see that

$$\|\psi * f\|_p \leqslant CK(1, f; H^{s_0}_p, H^{s_1}_p) \leqslant C\|f\|_{(H^{s_0}_p, H^{s_1}_p)_{\theta,q}}$$

and thus

$$\|f\|^s_{pq} \leqslant C\|f\|_{(H^{s_0}_p, H^{s_1}_p)_{\theta,q}}.$$

The other half of (10) follows easily from the inequalities (Lemma 6.2.1)

$$2^{k(s-s_0)} J(2^{k(s_1-s_0)}, \varphi_k * f; H^{s_0}_p, H^{s_1}_p) \leqslant C 2^{ks}\|\varphi_k * f\|_p,$$
$$J(1, \psi * f; H^{s_0}_p, H^{s_1}_p) \leqslant C\|\psi * f\|_p,$$

where $f \in B^s_{pq}$. It remains to show that

$$f = \psi * f + \textstyle\sum_{k=1}^{\infty} \varphi_k * f \quad \text{in} \quad H^{s_0}_p + H^{s_1}_p.$$

But if, say, $s_0 < s_1$ then $H^{s_0}_p + H^{s_1}_p = H^{s_0}_p$, and

$$\|\psi * f\|^{s_0}_p + \textstyle\sum_{k=1}^{\infty}\|\varphi_k * f\|^{s_0}_p \leqslant C(\|\psi * f\|_p + \textstyle\sum_{k=1}^{\infty} 2^{k(s_0-s)} 2^{ks}\|\varphi_k * f\|_p)$$
$$\leqslant C\|f\|^s_{pq},$$

by Hölder's inequality, since $s_0 < s$. □

The next theorem points to an alternative definition of the Besov spaces B^s_{pq} $(s>0)$ in terms of derivatives and moduli of continuity. The modulus of continuity is defined by

$$\omega^m_p(t, f) = \sup_{|y|<t}\|\Delta^m_y f\|_p,$$

where Δ_y^m is the m-th order difference operator:

$$\Delta_y^m f(x) = \sum_{k=0}^m \binom{m}{k} (-1)^k f(x+ky).$$

6.2.5. Theorem. *Assume that* $s>0$, *and let* m *and* N *be integers, such that* $m+N>s$ *and* $0 \leqslant N < s$. *Then, with* $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$,

$$\|f\|_{pq}^s \sim \|f\|_p + \sum_{j=1}^n (\int_0^\infty (t^{N-s} \omega_p^m(t, \partial^N f/\partial x_j^N))^q dt/t)^{1/q}.$$

Proof: We note that ω_p^m is an increasing function of t. Therefore it is sufficient to prove that

$$\|f\|_{pq}^s \sim \|f\|_p + \sum_{j=1}^n (\sum_{i=-\infty}^\infty (2^{i(s-N)} \omega_p^m(2^{-i}, \partial^N f/\partial x_j^N))^q)^{1/q}),$$

First we assume that $f \in B_{pq}^s$. Put $\hat{\rho}_y(\xi) = (1-\exp(i\langle y, \xi \rangle))^m$. We shall prove that for all integers k

$$\|\rho_y * \varphi_k * \partial^N f/\partial x_j^N\|_p \leqslant C \min(1, |y|^m 2^{mk}) 2^{Nk} \|\varphi_k * f\|_p, \tag{11}$$

and that

$$\|\rho_y * \psi * \partial^N f/\partial x_j^N\|_p \leqslant C \min(1, |y|^m) \|\psi * f\|_p. \tag{12}$$

Before proving these estimates, we shall show that they give the desired conclusion.

Thus suppose that (11) and (12) hold. Then clearly

$$\begin{aligned} &2^{i(s-N)} \omega_p^m(2^{-i}, \partial^N f/\partial x_j^N) \\ &\leqslant C(\sum_{k=1}^\infty 2^{(i-k)(s-N)} \min(1, 2^{-(i-k)m}) 2^{sk} \|\varphi_k * f\|_p + \min(1, 2^{-im}) \|\psi * f\|_p). \end{aligned}$$

The right hand side is a convolution of two sequences, namely the sequence $(2^{k(s-N)} \min(1, 2^{-km}))_{k=-\infty}^\infty$ and the sequence $(a_k)_{k=-\infty}^\infty$, where $a_k = 2^{sk} \|\varphi_k * f\|_p$ if $k \geqslant 1$, $a_0 = \|\psi * f\|_p$ and $a_k = 0$ if $k < 0$. Since

$$\sum_{k=-\infty}^\infty 2^{k(s-N)} \min(1, 2^{-km}) < \infty$$

we conclude that

$$(\sum_{i=-\infty}^\infty (2^{i(s-N)} \omega_p^m(2^{-i}, \partial^N f/\partial x_j^N))^q)^{1/q} \leqslant C(\sum_{k=-\infty}^\infty a_k^q)^{1/q} = C \|f\|_{pq}^s,$$

In order to prove (11) we note that $\hat{\rho}_y \in M_1$ and $\hat{\rho}_y(\cdot)\langle y, \cdot \rangle^{-m} \in M_1$ and

$$\|\hat{\rho}_y\|_{M_1} \leqslant C$$

$$\|\hat{\rho}_y(\cdot)\langle y, \cdot \rangle^{-m}\|_{M_1} \leqslant C$$

for all $y \neq 0$. This follows from Lemma 6.1.5 and Theorem 6.1.3. Similarly

$$\|\langle y/|y|, \cdot\rangle^m \varphi(\cdot)\|_{M_1} \leqslant C,$$

which implies

$$\|\langle y, \cdot\rangle^m \varphi(2^{-k}\cdot)\|_{M_1} \leqslant C\,|y|^m 2^{mk}.$$

It follows that

$$\begin{aligned}\|\rho_y * \varphi_k * \partial^N f/\partial x_j^N\|_p &\leqslant C\min(1, |y|^m 2^{mk}) \|\varphi_k * \partial^N f/\partial x_j^N\|_p \\ &\leqslant C\min(1, |y|^m 2^{mk})\, 2^{Nk}\, \|\varphi_k * f\|_p .\end{aligned}$$

This proves (11). The estimate (12) follows in the same way.

The converse inequality will follow if we can prove the estimate

$$\|\varphi_k * f\|_p \leqslant C 2^{-Nk} \textstyle\sum_{j=1}^n \|\rho_{jk} * \partial^N f/\partial x_j^N\|_p \tag{13}$$

where $\rho_{jk} = \rho_{(2^{-k}e_j)}$, e_j being the unit vector in the direction of the ξ_j-axis. In fact, if (13) is valid we have (since $\psi \in M_1$)

$$\begin{aligned}\|f\|_{pq}^s &\leqslant C(\|f\|_p + (\textstyle\sum_{k=1}^\infty (2^{k(s-N)} \sum_{j=1}^n \|\rho_{jk} * \partial^N f/\partial x_j^N\|_p)^q)^{1/q}) \\ &\leqslant C(\|f\|_p + \textstyle\sum_{j=1}^n (\sum_{k=1}^\infty (2^{k(s-N)}\ \omega_p^m(2^{-k}, \partial^N f/\partial x_j^N))^q)^{1/q}),\end{aligned}$$

which implies the desired inequality.

In order to prove (13) we need the following lemma.

6.2.6. Lemma. *Assume that* $n \geqslant 2$ *and take* φ *as in Lemma* 6.1.7. *Then there exist functions* $\chi_j \in \mathscr{S}(\mathbb{R}^n)$ $(1 \leqslant j \leqslant n)$, *such that*

$$\textstyle\sum_{j=1}^n \hat{\chi}_j = 1 \quad \textit{on} \quad \operatorname{supp}\varphi = \{\xi \,|\, 2^{-1} \leqslant |\xi| \leqslant 2\}$$
$$\operatorname{supp}\hat{\chi}_j \subset \{\xi \in \mathbb{R}^n \,|\, |\xi_j| \geqslant (3\sqrt{n})^{-1}\} \qquad (1 \leqslant j \leqslant n).$$

Proof of the lemma: Choose $k \in \mathscr{S}(\mathbb{R})$ with $\operatorname{supp} k = \{\xi \in \mathbb{R} \,|\, |\xi| \geqslant (3\sqrt{n})^{-1}\}$ and with positive values in the interior of supp k. Moreover, choose $l \in \mathscr{S}(\mathbb{R}^{n-1})$ with $\operatorname{supp} l = \{\xi \in \mathbb{R}^{n-1} \,|\, |\xi| \leqslant 3\}$ and positive in the interior. Writing $\check{\xi}^j = (\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n)$ and putting

$$\hat{\chi}_j(\xi) = k(\xi_j)\, l(\check{\xi}^j) / \textstyle\sum_{j=1}^n k(\xi_j)\, l(\check{\xi}^j) \qquad (1 \leqslant j \leqslant n),$$

where $\sum_{j=1}^n k(\xi_j) l(\check{\xi}^j) > 0$ on supp φ, only routine verifications remain to complete the proof of the lemma. □

We now complete the proof Theorem 6.2.5, i.e. we prove Formula (13). By the previous lemma we obtain the formula

$$\begin{aligned}\|\varphi_k * f\|_p &\leqslant \|\mathscr{F}^{-1}\{\varphi(2^k\cdot)\mathscr{F}f\}\|_p \\ &\leqslant \textstyle\sum_{j=1}^n \|\mathscr{F}^{-1}\{\hat{\chi}_j(2^{-k}\cdot)\varphi(2^{-k}\cdot)\mathscr{F}f\}\|_p \\ &\leqslant C2^{-kN}\textstyle\sum_{j=1}^n \|\rho_{kj} * \partial^N f/\partial x_j^N\|_p,\end{aligned}$$

since, by Theorem 6.1.3 and Lemma 6.1.5, we have

$$\hat{\chi}_j(\xi)\varphi(\xi)(\exp(i\xi_j)-1)^{-m}\xi_j^{-N} \in M_p$$

for $1 \leqslant j \leqslant n, 1 \leqslant p \leqslant \infty$. □

We also have the following consequence of Lemma 6.2.1.

6.2.7. Theorem. *J^σ is an isomorphism between B^s_{pq} and $B^{s-\sigma}_{pq}$, and between H^s_p and $H^{s-\sigma}_p$.*

Proof: Obvious, in view of Lemma 6.2.1. □

6.2.8. Corollary. *If $1 \leqslant p < \infty$ we have*

$$(H^s_p)' = H^{-s}_{p'} \qquad (s \in \mathbb{R}).$$

If, in addition, $1 \leqslant q < \infty$ we also have

$$(B^s_{pq})' = B^{-s}_{p'q'} \qquad (s \in \mathbb{R}).$$

Proof: The first formula follows from Theorem 6.2.7 and the fact that $(L_p)' = L_{p'}$ if $1 \leqslant p < \infty$. The second formula is implied by the first one, Theorem 6.2.4 and the duality theorem 3.8.1. □

6.3. The Homogeneous Sobolev and Besov Spaces

Sometimes it is convenient to work with symmetric sums of the form

$$\|f\|^{\dot{}s}_{pq} = (\textstyle\sum_{k=-\infty}^{\infty} (2^{sk}\|\varphi_k * f\|_p)^q)^{1/q},$$

where φ_k are the function defined in the previous section and $f \in \mathscr{S}'$. The space of all $f \in \mathscr{S}'$ for which $\|f\|^{\dot{}s}_{pq}$ is finite will be denoted by $\dot{B}^s_{pq}$ *homogeneous Besov space*. Note that $\dot{B}^s_{pq}$ is a semi-normed space and that $\|f\|^{\dot{}s}_{pq} = 0$ if and only if $\operatorname{supp}\hat{f} = \{0\}$, i.e. if and only if f is a polynomial.

There is also an analogous notion of a *homogeneous (generalized) Sobolev space* $\dot{H}^s_p$. The elements of $\dot{H}^s_p$ are those $f \in \mathscr{S}'$ for which $\sum_{-\infty}^{\infty} \mathscr{F}^{-1}(|\xi|^s \mathscr{F} \varphi_k * f)$ converges in $\mathscr{S}'$ to an L_p-function. We write

$$\|f\|^{\dot{}s}_p = \|\sum_{k=-\infty}^{\infty} \mathscr{F}^{-1}(|\xi|^s \mathscr{F} \varphi_k * f)\|_p.$$

(Note that $|\xi|^s \mathscr{F}(\varphi_k * f)$ is always a tempered distribution.) Again $\dot{H}^s_p$ is a semi-normed space and $\|f\|^{\dot{}s}_p = 0$ if and only if f is a polynomial.

Several results for the spaces B^s_{pq} and H^s_p carry over to the spaces $\dot{B}^s_{pq}$ and $\dot{H}^s_p$. For instance we have the following theorem, which corresponds to Theorem 6.2.4 and Theorem 6.2.5.

6.3.1. Theorem. *If* $1 \leqslant q_1 < q_2 \leqslant \infty$ *we have*

$$\dot{B}^s_{pq_1} \subset \dot{B}^s_{pq_2} \qquad (s \in \mathbb{R}, 1 \leqslant p \leqslant \infty).$$

Moreover,

$$\dot{B}^s_{p1} \subset \dot{H}^s_p \subset \dot{B}^s_{p\infty} (s \in \mathbb{R}, 1 \leqslant p \leqslant \infty)$$

and

$$(\dot{H}^{s_0}_p, \dot{H}^{s_1}_p)_{\theta,q} = \dot{B}^s_{pq} \qquad (s = (1-\theta)s_0 + \theta s_1, 0 < \theta < 1, 1 \leqslant p, q \leqslant \infty).$$

For $s > 0$, *and* m, N *integers such that* $m + N > s$ *and* $0 \leqslant N < s$ *we have*

$$\|f\|^{\dot{}s}_{pq} \sim \sum_{j=1}^{N} (\int_0^\infty (t^{N-s} \omega_p^m(t, \partial^N f/\partial x_j^N))^q dt/t)^{1/q} \qquad (1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty).$$

Finally, for any positive integer N,

$$\|f\|^{\dot{}N}_p \sim \sum_{j=1}^{n} \|\partial^N f/\partial x_j^N\|_p \qquad (1 < p < \infty),$$

if $\hat{f}$ *vanishes in a neighbourhood of the origin.*

Proof: The first inclusion is obvious. In order to prove that $\dot{H}^s_p \subset \dot{B}^s_{p\infty}$, let $f \in \dot{H}^s_p$. Then $\sum_{-\infty}^{\infty} \mathscr{F}^{-1}(|\xi|^s \mathscr{F} \varphi_k * f)$ converges in $\mathscr{S}'$ to a function $g \in L_p$. Define χ_k by the formula $\mathscr{F}\chi_k(\xi) = |\xi|^{-s} \varphi(2^{-k}\xi)$. Then, for any $h \in \mathscr{S}$,

$$\varphi_k * f * h(0) = \sum_{l=-\infty}^{\infty} \varphi_l * f * \varphi_k * h(0) = g * \chi_k * h(0).$$

Since $\|\mathscr{F}\chi_k\|_{M_{p'}} \leqslant C 2^{-sk}$ we conclude that

$$|\varphi_k * f * h(0)| \leqslant C 2^{-sk} \|g\|_p \|h\|_{p'}.$$

It follows that $\|\varphi_k * f\|_{L_p} \leqslant C 2^{-sk}$, i.e. $f \in \dot{B}^s_{p\infty}$.

Next we prove that $B^s_{p1} \subset \dot{H}^s_p$. Take $f \in \dot{B}^s_{p1}$. Then easily

$$\|\mathscr{F}^{-1}\{|\xi|^s \mathscr{F}\varphi_k * f\}\|_p \leqslant C2^{sk}\|\varphi_k * f\|_p.$$

Therefore $\sum_{-\infty}^{\infty} \mathscr{F}^{-1}(|\xi|^s \mathscr{F}\varphi_k * f)$ converges in L_p and thus $f \in \dot{H}^s_p$.

The proof of the interpolation result is the same as that of Formula (10) in Theorem 6.2.4.

The equivalent representation of the semi-norm on $\dot{B}^s_{pq}$ follows at once from the proof of Theorem 6.2.5 (see Formula (11) and (13)).

For $f \in \mathscr{S}$ we have

$$\partial^N f/dx_j^N = \sum_{k=-\infty}^{\infty} \partial^N \varphi_k * f/\partial x_j^N = \mathscr{F}^{-1}(\xi_j^N |\xi|^{-N} \sum_{k=-\infty}^{\infty} |\xi|^N \mathscr{F}\varphi_k * f)$$

(with convergence in L_p) and since $\xi_j^N |\xi|^{-N} \in M_p$ for $1 < p < \infty$, we obtain

$$\|\partial^N f/\partial x_j^N\|_p \leqslant C\|f\|^{\dot{}N}_p.$$

Conversely, let ψ be defined as in Section 6.2 and let χ be as in the proof of Theorem 6.2.3. Then

$$|\xi|^N \hat{\psi}(\xi) \cdot (\sum_{j=1}^n \chi(\xi_j)|\xi_j|^N)^{-1} \in M_p.$$

Thus if $\hat{f}(\xi) = 0$ for $|\xi| \leqslant 2\delta$,

$$\begin{aligned}\|\mathscr{F}^{-1}\{|\xi|^N \mathscr{F} f\}\|_p &= \delta^N \|\mathscr{F}^{-1}\{(|\xi|/\delta)^N \hat{\psi}(\xi/\delta) \mathscr{F} f\}\|_p \\ &\leqslant C\delta^N \sum_{j=1}^n \|\mathscr{F}^{-1}\{\chi(\xi_j/\delta)(|\xi_j|/\delta)^N \mathscr{F} f\}\|_p \\ &\leqslant C \sum_{j=1}^n \|\partial^N f/\partial x_j^N\|_p,\end{aligned}$$

since $\chi(\xi_j)|\xi_j|^N \xi_j^{-N} \in M_p$. □

Next we investigate the connection between the dotted and the non-dotted spaces.

6.3.2. Theorem. *Suppose that* $f \in \mathscr{S}'$ *and* $0 \notin \operatorname{supp} \hat{f}$. *Then*

$$f \in \dot{B}^s_{pq} \Leftrightarrow f \in B^s_{pq} \quad (s \in \mathbb{R}, 1 \leqslant p, q \leqslant \infty),$$

$$f \in \dot{H}^s_p \Leftrightarrow f \in H^s_p \quad (s \in \mathbb{R}, 1 \leqslant p \leqslant \infty).$$

Moreover,

(1) $$B^s_{pq} = L_p \cap \dot{B}^s_{pq} \quad (s > 0, 1 \leqslant p, q \leqslant \infty),$$

(2) $$H^s_p = L_p \cap \dot{H}^s_p \quad (s > 0, 1 \leqslant p \leqslant \infty).$$

Proof: If $\hat{f}(\xi)=0$ in a neighbourhood of $\xi=0$ and if $f\in\dot{B}^s_{pq}$, then $\psi*f$ (being a finite sum of the form $\sum_k\psi*\varphi_k*f$) belongs to L_p. Thus $f\in B^s_{pq}$. Conversely, if $f\in B^s_{pq}$ then $\varphi_k*f=\varphi_k*\psi*f$ if $k<0$. Thus $\varphi_k*f\in L_p$ for all k and, since $(\sum_{k<0}(2^{sk}\|\varphi_k*f\|_p)^q)^{1/q}$ is a finite sum, $f\in\dot{B}^s_{pq}$.

If still $\hat{f}(\xi)=0$ in a neighbourhood of $\xi=0$ then $f=\sum_{k\geq k_0}\varphi_k*f$ for some integer k_0. Noting that $(1+|\xi|^2)^{s/2}|\xi|^{-s}\sum_{l\geq k_0-1}\varphi(2^{-l}\xi)\in M_1$ (see Exercise 3 or 4), we see that for $f\in\dot{H}^s_p$

$$\|f\|^s_p=\|\mathscr{F}^{-1}\{(1+|\xi|^2)^{s/2}|\xi|^{-s}\textstyle\sum_{l\geq k_0-1}\varphi(2^{-l}\xi)\sum_{k\geq k_0}|\xi|^s\mathscr{F}\varphi_k*f\}\|_p$$
$$\leq C\|f\dot{\|}^s_p.$$

Conversely, if $f\in H^s_p$, then we note that $|\xi|^s(1+|\xi|^2)^{-s/2}\in M_1$ (see Exercise 3 or 4). Thus

$$\|f\dot{\|}^s_p=\|\mathscr{F}^{-1}\{|\xi|^s(1+|\xi|^2)^{-s/2}\mathscr{F}J^sf\}\|_p\leq C\|f\|^s_p.$$

(This holds without the assumption $\hat{f}(\xi)=0$ in a neighbourhood of $\xi=0$.)

In order to prove (1) we first note that it is obvious that for all $s\in\mathbb{R}$

$$L_p\cap\dot{B}^s_{pq}\subset B^s_{pq}.$$

Conversely, if $f\in B^s_{pq}$, then $\|\varphi_k*f\|_p\leq C\|\psi*f\|_p$ for $k<0$. Thus if $s>0$

$$(\textstyle\sum_{k<0}(2^{sk}\|\varphi_k*f\|_p)^q)^{1/q}\leq C\|\psi*f\|_p,$$

then $f\in\dot{B}^s_{pq}$.

If $f\in L_p\cap\dot{H}^s_p$, then we obtain as above

$$\|f\|^s_p\leq\|\mathscr{F}^{-1}\{(1+|\xi|^2)^{s/2}|\xi|^{-s}\textstyle\sum_{l\geq0}\varphi(2^{-l}\xi)\sum_{k\geq0}|\xi|^s\mathscr{F}\varphi_k*f\}\|_p$$
$$+\|\mathscr{F}^{-1}\{(1+|\xi|^2)^{s/2}\textstyle\sum_{k<0}\varphi(2^{-k}\xi)\mathscr{F}f\}\|_p\leq C(\|f\dot{\|}^s_p+\|f\|_p).$$

(This holds for all $s\in\mathbb{R}$.) Conversely if $s>0$ and $f\in H^s_p$, then clearly $f\in L_p$ and

$$\|f\dot{\|}^s_p\leq\|\mathscr{F}^{-1}\{|\xi|^s(1+|\xi|^2)^{-s/2}\mathscr{F}\textstyle\sum_{k\geq1}\varphi_k*J^sf\}\|_p$$
$$+\textstyle\sum_{k\leq0}2^{sk}\|\mathscr{F}^{-1}\{(2^{-k}|\xi|)^s\varphi(2^{-k}\xi)\mathscr{F}f\}\|_p$$
$$\leq C(\|\textstyle\sum_{k\geq1}\varphi_k*f\|^s_p+\|f\|_p)\leq C\|f\|^s_p.\quad\square$$

6.4. Interpolation of Sobolev and Besov Spaces

We have already established that if $s_0\neq s_1$ then (Theorem 6.2.4)

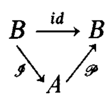
$$(H^{s_0}_p,H^{s_1}_p)_{\theta,q}=B^s_{pq},\qquad(0<\theta<1,1\leq p,q\leq\infty),$$

where $s=(1-\theta)s_0+\theta s_1$. We shall now prove some other interpolation results using theorems from Chapter 5 on interpolation of L_p spaces. To bring out the connection we introduce the concept of retract. For other terminology, see Chapter 2.

The results throughout this section are stated for the non-homogeneous spaces H_p^s and B_{pq}^s. However, it is easy to adapt the proofs so that all the theorems hold when the homogeneous spaces $\dot{H}_p^s$ and $\dot{B}_{pq}^s$ are substituted for H_p^s and B_{pq}^s respectively. (Cf. Notes and Comment.)

6.4.1. Definition. *An object B, in a given category, is called a retract of the object A, if there are morphisms $\mathscr{I}:B\to A$ and $\mathscr{P}:A\to B$, in the category, such that $\mathscr{P}\circ\mathscr{I}$ is the identity on B.*

If B is a retract of A we have the following commutative diagram



the letters $\mathscr{I}$ and $\mathscr{P}$ being used to remind the reader of the words injection and projection.

6.4.2. Theorem. *Assume that $\bar{B}$ is a retract of $\bar{A}$ in the category $\mathscr{N}_1$ (of all compatible couples of normed linear spaces), with mappings $\mathscr{I}$ and $\mathscr{P}$. Then $\bar{B}_{[\theta]}$ and $\bar{B}_{\theta,q}$ are retracts in $\mathscr{N}$ of $\bar{A}_{[\theta]}$ and $\bar{A}_{\theta,q}$ respectively.*

Proof: The theorem follows at once from the interpolation properties. □

We shall now introduce two mappings $\mathscr{I}$ and $\mathscr{P}$. The mapping $\mathscr{I}$ maps $\mathscr{S}'$ to the space of all sequences of tempered distributions. It is defined by

$$(1)\qquad \begin{aligned}&(\mathscr{I}f)_j=\varphi_j*f \quad\text{for}\quad j=1,2,\dots,\\ &(\mathscr{I}f)_0=\psi*f.\end{aligned}$$

The mapping $\mathscr{P}$ is given by

$$(2)\qquad \mathscr{P}\alpha=\textstyle\sum_{j=0}^{\infty}\tilde{\varphi}_j*\alpha_j,$$

where $\alpha=(\alpha_j)_{j=0}^{\infty}$, $\alpha_j\in\mathscr{S}'$, $j=0,1,\dots$ and

$$\tilde{\varphi}_0=\psi+\varphi_1,$$
$$\tilde{\varphi}_j=\textstyle\sum_{l=-1}^{1}\varphi_{j+l},\qquad j=1,2,\dots.$$

We are not saying that $\mathscr{P}$ is defined on all sequences $(\alpha_j)_{j=0}^{\infty}$ of tempered distributions, but only on those sequences for which the series defining $\mathscr{P}\alpha$ converges in $\mathscr{S}'$. Clearly $\mathscr{P}\mathscr{I}f=f$, $f\in\mathscr{S}'$, since $\tilde{\varphi}_j*\varphi_j=\varphi_j$, $j=1,2,3,\dots$ and $\tilde{\varphi}_0*\psi=\psi$.

6.4.3. Theorem. *The space* B^s_{pq} *is a retract of* $l^s_q(L_p)$ $(s\in\mathbb{R}, 1\leq p,q\leq\infty)$, *and* H^s_p *is a retract of* $L_p(l^s_2)$, $(s\in\mathbb{R}, 1<p<\infty)$. *The mappings* $\mathscr{P}$ *and* $\mathscr{I}$ *are defined by* (1) *and* (2).

Proof: First we note that

$$\|f\|^s_{pq}\sim(\|\psi*f\|^q_p+\textstyle\sum_{j=1}^\infty(2^{js}\|\varphi_j*f\|_p)^q)^{1/q}=(\sum_{j=0}^\infty(2^{js}\|(\mathscr{I}f)_j\|_p)^q)^{1/q},$$

i.e.

$$\|f\|^s_{pq}\sim\|\mathscr{I}f\|_{l^s_q(L_p)}.$$

Thus $\mathscr{I}: B^s_{pq}\to l^s_q(L_p)$. Moreover, since $\psi*\mathscr{P}\alpha=\psi*\alpha_0$ and $\varphi_k*\mathscr{P}\alpha=\varphi_k*\alpha_k$ for $k\geq 1$, we have

$$\begin{aligned}\|\mathscr{P}\alpha\|^s_{pq}&=\|\psi*\alpha_0\|_p+(\textstyle\sum_{k\geq1}(2^{ks}\|\varphi_k*\alpha_k\|_p)^q)^{1/q}\\ &\leq C(\|\alpha_0\|_p+(\textstyle\sum_{k\geq1}(2^{ks}\|\alpha_k\|_p)^q)^{1/q})\leq C\|\alpha\|_{l^s_q(L_p)}.\end{aligned}$$

Thus B^s_{pq} is a retract of $l^s_q(L_p)$.

Next we prove that $\mathscr{I}: H^s_p\to L_p(l^s_2)$, $1<p<\infty$. We may write

$$\mathscr{I}f=(\mathscr{F}^{-1}\chi)*J^s f$$

where $\chi\in\mathscr{S}'(\mathbb{C},l^s_2)=L(\mathscr{S}(\mathbb{R}^n,\mathbb{C}),l^s_2)$ is defined by

$$(\chi(\xi))_j=(1+|\xi|^2)^{-s/2}\hat{\varphi}_j(\xi),\qquad j=1,2,\dots,$$

$$(\chi(\xi))_0=(1+|\xi|^2)^{-s/2}\hat{\psi}(\xi).$$

Then

$$|\xi|^{|\alpha|}\|D^\alpha\chi(\xi)\|_{L(\mathbb{C},l^s_2)}\leq|\xi|^{|\alpha|}(\textstyle\sum_{j=0}^\infty(2^{js}|D^\alpha(\chi(\xi))_j|)^2)^{1/2}\leq C_\alpha,$$

since the sum consists of at most two terms for each ξ. Thus $\mathscr{I}: H^s_p\to L_p(l^s_2)$ by Mihlin's multiplier theorem $(1<p<\infty)$.

Finally, we establish that $\mathscr{P}: L_p(l^s_2)\to H^s_p$, or, which is equivalent, that $J^s\mathscr{P}: L_p(l^s_2)\to L_p$. We may write

$$J^s\mathscr{P}\beta=\mathscr{F}^{-1}\kappa*\beta^s$$

where $\beta=(\beta_j)_{j=0}^\infty$ and $\beta^s=(2^{js}\beta_j)_{j=0}^\infty$, and

$$\kappa(\xi)\beta=\textstyle\sum_{j=0}^\infty 2^{-js}(1+|\xi|^2)^{s/2}\tilde{\varphi}_j(\xi)\beta_j.$$

Clearly, $\kappa\in\mathscr{S}'(l_2,\mathbb{C})=L(\mathscr{S}(\mathbb{R}^n;l_2),\mathbb{C})$ and

$$|\xi|^{|\alpha|}\|D^\alpha\kappa(\xi)\|_{L(l_2,\mathbb{C})}\leq|\xi|^{|\alpha|}(\textstyle\sum_{j=0}^\infty(2^{-js}|D^\alpha(1+|\xi|^2)^{s/2}\tilde{\varphi}_j(\xi)|)^2)^{1/2}\leq C_\alpha$$

since the sum consists of at most four terms for each ξ. By Mihlin's multiplier theorem it follows that $J^s\mathscr{P}: L_p(l_2^s)\to L_p$, $1<p<\infty$. The proof is complete. □

We also note the following important inclusion theorem.

6.4.4. Theorem. *We have the inclusions*

$$B^s_{pp}\subset H^s_p\subset B^s_{p2}, \qquad (s\in\mathbb{R}, 1<p\leqslant 2),$$

$$B^s_{p2}\subset H^s_p\subset B^s_{pp}, \qquad (s\in\mathbb{R}, 2\leqslant p<\infty).$$

Proof: By Theorem 6.2.7, we need consider only the case $s=0$. Let $1<p\leqslant 2$ and take $f\in B^0_{pp}$. Since $\mathscr{P}: L_p(l_2)\to L_p$ and $l_p\subset l_2$ we obtain

$$\begin{aligned}\|f\|_p&=\|\mathscr{P}\mathscr{I}f\|_p\leqslant C\|\mathscr{I}f\|_{L_p(l_2)}\leqslant C\|\mathscr{I}f\|_{L_p(l_p)}\\&=C\left(\int_{\mathbb{R}^n}\sum_{j\geqslant 0}|(\mathscr{I}f)_j(x)|^p dx\right)^{1/p}\leqslant C\|\mathscr{I}f\|_{l_p(L_p)}\leqslant C\|f\|^0_{pp}.\end{aligned}$$

Thus $B^0_{pp}\subset L_p$ for $1<p\leqslant 2$.

Next we prove that $L_p\subset B^0_{p2}$ for $1<p\leqslant 2$. Using Minkowski's inequality and the fact that $\mathscr{I}: L_p\to L_p(l_2)$ and $\mathscr{P}: l_2^0(L_p)\to B^0_{p2}$ we see that

$$\begin{aligned}\|f\|^0_{p2}&=\|\mathscr{P}\mathscr{I}f\|^0_{p2}\leqslant C\|\mathscr{I}f\|_{l_2^0(L_p)}=C\left(\sum_{j\geqslant 0}\left(\int_{\mathbb{R}^n}|(\mathscr{I}f)_j(x)|^p dx\right)^{2/p}\right)^{1/2}\\&\leqslant C\left(\int_{\mathbb{R}^n}\left(\sum_{j\geqslant 0}|(\mathscr{I}f)_j(x)|^2\right)^{p/2}dx\right)^{1/p}=C\|\mathscr{I}f\|_{L_p(l_2)}\leqslant C\|f\|_p.\end{aligned}$$

The case $2\leqslant p<\infty$ is settled by means of Corollary 6.2.8 and what was proved above. □

The following theorem, which is a consequence of Theorem 6.4.3 and the theorems of Chapter 5, contains the main interpolation results for generalized Sobolev and Besov spaces.

6.4.5. Theorem. *Let θ be given so that $0<\theta<1$. Moreover, let $s, s_0, s_1, p, p_0, p_1, q, q_0, q_1$ and r be given numbers subject to the restrictions given in the formulas below. In addition, put*

$$s^*=(1-\theta)s_0+\theta s_1,$$

$$\frac{1}{p^*}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},$$

$$\frac{1}{q^*}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

Then we have

(1) $$(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,r}=B^{s^*}_{pr}, \qquad (s_0\neq s_1, 1\leqslant p\leqslant\infty, 1\leqslant r, q_0, q_1\leqslant\infty),$$

(2) $(B^s_{pq_0}, B^s_{pq_1})_{\theta,q^*} = B^s_{pq^*}$, $(1 \leqslant p, q_0, q_1 \leqslant \infty)$,

(3) $(B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1})_{\theta,p^*} = B^{s^*}_{p^*q^*}$, $(s_0 \neq s_1, p^* = q^*, 1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty)$,

(4) $(H^{s_0}_p, H^{s_1}_p)_{\theta,q} = B^{s^*}_{pq}$, $(s_0 \neq s_1, 1 \leqslant p, q \leqslant \infty)$,

(5) $(H^s_{p_0}, H^s_{p_1})_{\theta,p^*} = H^s_{p^*}$, $(1 \leqslant p_0, p_1 \leqslant \infty)$,

(6) $(B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1})_{[\theta]} = B^{s^*}_{p^*q^*}$, $(s_0 \neq s_1, 1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty)$,

(7) $(H^{s_0}_{p_0}, H^{s_1}_{p_1})_{[\theta]} = H^{s^*}_{p^*}$, $(s_0 \neq s_1, 1 < p_0, p_1 < \infty)$.

Proof: The first two formulas follow from Theorem 6.4.3 and Theorem 5.7.1. Formula (3) follows from Theorem 6.4.3 and 5.7.2. Formula (4) is contained in Theorem 6.2.4, while Theorem 6.2.7 and Theorem 5.2.4 imply (5). Finally, (6) and (7) follow from Theorem 6.4.3 and Theorem 5.6.3 and Theorem 5.1.2. □

6.5. An Embedding Theorem

Consider the space

$$W^N_p = \{f \in \mathscr{S}' \mid \|f\|_{W^N_p} < \infty\},$$

where N is a positive integer and

$$\|f\|_{W^N_p} = \sum_{|\alpha| \leqslant N} \|D^\alpha f\|_p \quad (1 \leqslant p \leqslant \infty).$$

This space is the one originally defined by Sobolev. In Theorem 6.2.4, we state that $H^N_p = W^N_p$ $(1 < p < \infty)$.

It is well known that

$$W^N_p \subset L_{p_1} \quad (0 < n/p - N \leqslant n/p_1),$$

which is the Sobolev embedding theorem. In this section we shall prove a corresponding theorem for the spaces H^s_p and B^s_{pq}. We remark that, as in the previous section, all results are valid also for the homogeneous spaces $\dot{H}^s_p$ and $\dot{B}^s_{pq}$, the proofs being easily modified to cover these cases.

6.5.1. Theorem (The embedding theorem). *Assume that* $s - n/p = s_1 - n/p_1$. *Then the following inclusions hold*

$$B^s_{pq} \subset B^{s_1}_{p_1q_1} \quad (1 \leqslant p \leqslant p_1 \leqslant \infty, 1 \leqslant q \leqslant q_1 \leqslant \infty, s, s_1 \in \mathbb{R}),$$
$$H^s_p \subset H^{s_1}_{p_1} \quad (1 < p \leqslant p_1 < \infty, s, s_1 \in \mathbb{R}).$$

Proof: In order to prove the first inclusion, we use the estimates

$$\|\varphi_k*f\|_{p_1}=\|\tilde{\varphi}_k*\varphi_k*f\|_{p_1}\leq C2^{kn(1/p-1/p_1)}\|\varphi_k*f\|_p \qquad (k\geq 1).$$

They are immediate consequences of Young's inequality, Theorem 1.2.2, and the fact that $\|\tilde{\varphi}_k\|_q\leq C2^{kn(1-1/q)}$ $(k\geq 0)$. Similarly we see that

$$\|\psi*f\|_{p_1}\leq C\|\psi*f\|_p.$$

By Theorem 6.2.4 we therefore infer that

$$\begin{aligned}\|f\|^{s_1}_{p_1q_1}&=\|\psi*f\|_{p_1}+(\textstyle\sum_{k\geq 1}(2^{ks_1}\|\varphi_k*f\|_{p_1})^{q_1})^{1/q_1}\\&\leq C(\|\psi*f\|_p+(\textstyle\sum_{k\geq 1}(2^{ks}\|\varphi_k*f\|_p)^{q_1})^{1/q_1})\leq C\|f\|^s_{pq},\end{aligned}$$

since $q\leq q_1$. This gives the first inclusion.

The second inclusion is proved using interpolation, with the aid of the first one. Clearly, we need only consider the case $s_1=0$, in view of Theorem 6.2.7. Invoking Theorem 6.2.4, we have the inclusions

$$B^s_{p1}\subset B^0_{p_11}\subset H^0_{p_1}=L_{p_1}.$$

Interpolating (with fixed p) the inclusions

$$\begin{aligned}&B^{s'}_{p1}\subset L_{p'_1} \qquad (s'-n/p=-n/p'_1),\\&B^{s''}_{p1}\subset L_{p''_1} \qquad (s''-n/p=-n/p''_1),\end{aligned}$$

we obtain (Theorem 6.4.5)

$$B^s_{p\infty}=(B^{s'}_{p1},B^{s''}_{p1})_{\theta,\infty}\subset(L_{p'_1},L_{p''_1})_{\theta,\infty}=L_{p_1\infty},$$

where θ is chosen appropriately. It follows that

$$H^s_p\subset L_{p_1\infty},$$

again by Theorem 6.2.4. Interpolating (with fixed s) the inclusions

$$\begin{aligned}&H^s_{q'}\subset L_{q'_1\infty} \qquad (s-n/q'=-n/q'_1),\\&H^s_{q''}\subset L_{q''_1\infty} \qquad (s-n/q''=-n/q''_1),\end{aligned}$$

we obtain (Theorem 6.4.5, 5.3.1, $p\leq p_1$)

$$H^s_p=(H^s_{q'},H^s_{q''})_{\theta,p}\subset(L_{q'_1\infty},L_{q''_1\infty})_{\theta,p}=L_{p_1p}\subset L_{p_1},$$

where, again, θ is chosen suitably. □

6.6. A Trace Theorem

In this section we use the notation $B^s_{pq}(\mathbb{R}^n)$ for the Besov space on $\mathbb{R}^n$. The spaces $H^s_p(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ and so on are defined analogously. We shall consider the trace operator

$$\mathrm{Tr}\colon \mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^{n-1}),$$

defined by

$$(\mathrm{Tr}f)(x')=f(0,x'), \qquad x'=(x_2,\ldots,x_r).$$

We shall prove the following result.

6.6.1. Theorem (The trace theorem). *Assume that* $1<p<\infty$, $1\leqslant q\leqslant\infty$, $s>1/p$. *Then the trace operator can be extended so that*

$$\mathrm{Tr}\colon B^s_{pq}(\mathbb{R}^n)\to B^{s-1/p}_{pq}(\mathbb{R}^{n-1}),$$
$$\mathrm{Tr}\colon H^s_p(\mathbb{R}^n)\to B^{s-1/p}_{pp}(\mathbb{R}^{n-1}).$$

Proof: We shall prove that

$$\text{(1)} \qquad \mathrm{Tr}\colon H^m_p(\mathbb{R}^n)\to B^{m-1/p}_{pp}(\mathbb{R}^{n-1}), \qquad m=1,2,\ldots .$$

In view of Theorem 6.4.5 (Formulas (1), (4), (6) and (7)), this implies the theorem for $s>1$. By density we need consider only functions in $\mathcal{S}$.

Let $\mathbb{R}^n_+$ be the half-space $x_1>0$ and let $H^m_p(\mathbb{R}^n_+)$ denote the space of all $f\in L_p(\mathbb{R}^n_+)$ such that $\partial^m f/\partial x_j^m\in L_p(\mathbb{R}^n_+)$ for $j=1,\ldots,n$. The norm on $H^m_p(\mathbb{R}^n_+)$ is

$$\|f\|_{H^m_p(\mathbb{R}^n_+)}=\|f\|_{L_p(\mathbb{R}^n_+)}+\textstyle\sum_{j=1}^n\|\partial^m f/\partial x_j^m\|_{L_p(\mathbb{R}^n_+)}.$$

Let $\mathcal{R}$ be the restriction operator for $\mathbb{R}^n_+$. Then, by Theorem 6.3.3, $\mathcal{R}\colon H^m_p(\mathbb{R}^n)\to H^m_p(\mathbb{R}^n_+)$. Moreover $\mathrm{Tr}f=\mathrm{Tr}\,\mathcal{R}f$. Thus (1) will follow if we prove

$$\text{(2)} \qquad \mathrm{Tr}\colon H^m_p(\mathbb{R}^n_+)\to B^{m-1/p}_{pp}(\mathbb{R}^{n-1}).$$

Now put

$$A_0=H^m_p(\mathbb{R}^{n-1}), \qquad A_1=L_p(\mathbb{R}^{n-1}).$$

Then

$$B^{m-1/p}_{pp}(\mathbb{R}^{n-1})=\bar{A}_{\theta,p} \quad \text{if} \quad \theta=1/mp.$$

Put $u(t)=f(t,x')$ for $t>0$ if $f\in\mathcal{S}$. Clearly $u(t)\to\mathrm{Tr}f$ in $L_p(\mathbb{R}^{n-1})$ as $t\to 0$. Using Corollary 3.12.3 we see that

$$\|\mathrm{Tr}f\|_{\bar{A}_{\theta,p}}\leqslant C\max(\|t^{1/p}u(t)\|_{L^*_p(A_0)}, \|t^{1/p}u^{(m)}(t)\|_{L^*_p(A_1)}).$$

But the right hand side is equivalent to the norm of f in $H_p^m(\mathbb{R}_+^n)$. Thus (2) follows.

It remains to prove the theorem for $1/p < s \leqslant 1$. By the embedding theorem we know that $B_{pp}^{1/p}(\mathbb{R}) \subset L_\infty(\mathbb{R})$, so that

$$|\mathrm{Tr} f(x')| \leqslant C \| f(\cdot, x') \|_{B_{p1}^{1/p}(\mathbb{R})}$$

for all $x' \in \mathbb{R}^{n-1}$. Integrating over $\mathbb{R}^{n-1}$ and using Theorem 6.2.5 and Minkowski's inequality we deduce that

$$\| \mathrm{Tr} f \|_{L_p(\mathbb{R}^{n-1})} \leqslant C \| f \|_{B_{p1}^{1/p}(\mathbb{R})},$$

i.e.

$$\mathrm{Tr}: B_{p1}^{1/p}(\mathbb{R}^n) \to L_p(\mathbb{R}^{n-1}). \tag{3}$$

Using (1) and Theorem 6.4.5 this implies the theorem for $B_{pq}^s(\mathbb{R}^n)$, $s > 1/p$.

By Theorem 6.4.4 we know that

$$H_p^s(\mathbb{R}^n) \subset B_{pp}^s(\mathbb{R}^n), \qquad 2 \leqslant p < \infty.$$

Thus the theorem for $H_p^s(\mathbb{R}^n)$ follows from what we have already proved in the case $2 \leqslant p < \infty$.

From what we have already proved we also see that

$$\mathrm{Tr}: H_2^{s_0}(\mathbb{R}^n) \to B_{22}^{s_0 - 1/2}(\mathbb{R}^{n-1}), \qquad s_0 > 1/2,$$
$$\mathrm{Tr}: H_{p_1}^1(\mathbb{R}^n) \to B_{p_1 p_1}^{1-1/p_1}(\mathbb{R}^{n-1}), \qquad 1 < p_1 < \infty.$$

Thus by Theorem 6.4.5 (Formula (6) and (7)),

$$\mathrm{Tr}: H_p^s(\mathbb{R}^n) \to B_{pp}^{s-1/p}(\mathbb{R}^{n-1}), \tag{4}$$

if

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_1}, \qquad s = (1-\theta)s_0 + \theta.$$

But if $1 < p < 2$, $s > 1/p$ are given we can find p_1, $(1 < p_1 < p)$ so that s_0, given by the relations above, satisfies $s_0 > 1/2$. Thus (4) holds as soon as $1 < p < 2$, $s > 1/p$. □

6.7. Interpolation of Semi-Groups of Operators

A reader who is not familiar with semi-groups of operators might substitute the group of translations in $\mathbb{R}^n$ for the general semi-group in a first reading. In fact, the group of translations in $\mathbb{R}^n$ is, in a way, the generating case for the semi-group approach. After the definitions, we give an example to illustrate what semi-groups comprise. Together with an interpolation result, we use this example to obtain yet another characterization of the Besov spaces B^s_{pq}.

Let A be a Banach space, and let $\{G(t)\}$ $(t>0)$ be a family of bounded linear operators from A to A. Then we shall say that $\{G(t)\}$ is *an equi-bounded, strongly continuous semi-group of operators on* A if the following three conditions hold:

(i) $G(s+t)a=G(s)(G(t)a) \quad (s,t>0,\ a\in A)$,

(ii) $\|G(t)a\|_A \leqslant M\|a\|_A \quad (t>0,\ a\in A)$,

(iii) $\lim_{t\to+0}\|G(t)a-a\|_A=0 \quad (a\in A)$.

Routine verifications show that $t\to G(t)a$ is a strongly continuous function on $\mathbb{R}_+$.

The *infinitesimal generator* Λ of the semi-group $\{G(t)\}$ is defined by the formula

$$\lim_{t\to+0}\|t^{-1}(G(t)a-a)-\Lambda a\|_A=0.$$

The *domain* $D(\Lambda)$ of Λ is obviously the space of all $a\in A$, such that $\lim_{t\to+0}t^{-1}(G(t)a-a)$ exists. Also, Λ is a linear operator, and, in non-trivial cases, it is not bounded.

Example: We let A stand for any space among $L_p(\mathbb{R}^n, dx)$ $(1\leqslant p<\infty)$ or the closure of $\mathscr{S}$ in L_∞, the latter space consisting of all continuous functions which tend to zero at infinity.

H denotes an infinitely differentiable positive function on $\mathbb{R}^n\setminus\{0\}$, which is positively homogeneous of order $m>0$, i. e. $H(t\xi)=|t|^m H(\xi)$ $(t\in\mathbb{R},\ \xi\in\mathbb{R}^n\setminus\{0\})$.

The family $\{G(t)\}$, defined by

$$G(t)a=\mathscr{F}^{-1}\{\exp(-tH)\mathscr{F}a\} \quad (a\in\mathscr{S},\ t>0),$$

is a semi-group on A. To prove this, we have to verify that $G(t)$ are operators from A to A, and that the conditions (i)—(iii) are satisfied. First, we note that $\exp(-tH)\in M_p$ $(1\leqslant p\leqslant\infty,\ t>0)$, with a norm which does not depend on t, by Theorem 6.1.3, Lemma 6.1.5 and the homogeneity of H (cf. Exercise 4). It follows that $G(t)$ may be extended to a mapping from A to A (sic), and that (ii) holds, since its domain is dense in A. Clearly, (i) is also satisfied. To prove (iii), take $a\in\mathscr{S}$. Then we have

$$G(t)a-a=\int_0^t G(s)\mathscr{F}^{-1}(-H\mathscr{F}a)\,ds$$

and thus, using (ii), we obtain

$$\|G(t)a-a\|_A\leqslant M\cdot t\cdot\|\mathscr{F}^{-1}(H\mathscr{F}a)\|_A\to 0 \quad (t\to+0).$$

Since $\mathscr{S}$ is dense in A, (iii) follows by using (ii) once more.

It is clear that the infinitesimal generator of $G(t)$ is the operator Λ, defined by

$$\Lambda a=-\mathscr{F}^{-1}\{H\mathscr{F}a\} \quad (a\in\mathscr{S}).$$

More exactly, the infinitesimal generator of $G(t)$ is the closure in A of the operator Λ. This is a consequence of the next lemma (see below).

As particular cases of this example, we mention $H(\xi)=|\xi|^2$ and $H(\xi)=|\xi|$. If $H(\xi)=|\xi|^2$, the semi-group is given by

$$G(t)a(x)=(4\pi t)^{-n/2}\int_{\mathbb{R}^n}\exp(-|u|^2/4t)\,a(x-u)\,du$$

(see Butzer and Behrens [1] for details). This integral is usually called the Gauss-Weierstrass singular integral. If $H(\xi)=|\xi|$ the semi-group is defined by means of a convolution with the Poisson kernel for the half-space $\mathbb{R}^n\times\mathbb{R}_+$:

$$G(t)\,a(x)=\pi^{-(n+1)/2}\,\Gamma((n+1)/2)\int_{\mathbb{R}^n}t(t^2+|u|^2)^{-(n+1)/2}\,a(x-u)\,du$$

(see Butzer and Behrens [1]).

Now we return to the general situation. The following lemma displays frequently used properties.

6.7.2. Lemma. *Let $\{G(t)\}$ be an equi-bounded, strongly continuous semi-group of operators on A, with Λ as its infinitesimal generator. Then Λ is closed, and its domain $D(\Lambda)$ is a Banach space in the graph norm*

$$\|a\|_{D(\Lambda)}=\|a\|_A+\|\Lambda a\|_A \qquad (a\in D(\Lambda))\,.$$

In addition, if $a\in D(\Lambda)$ then $G(t)a\in D(\Lambda)$, and

$$(1) \qquad d(G(t)a)/dt=\Lambda G(t)a=G(t)\Lambda a\,,$$

$$(2) \qquad G(t)a-a=\int_0^t G(s)\Lambda a\,ds\,.$$

Finally, $D(\Lambda)$ is dense in A.

Proof: Formula (1) follows at once from the definitions. (2) follows from (1), since if $a'\in A'$ then

$$\begin{aligned}\langle G(t)a-a,a'\rangle&=\int_0^t(d\langle G(s)a,a'\rangle/ds)ds=\int_0^t\langle G(s)\Lambda a,a'\rangle ds\\&=\langle\int_0^t G(s)\Lambda a\,ds,a'\rangle\,.\end{aligned}$$

To prove the density of $D(\Lambda)$ in A, take $a\in A$. Then we have, by Formula (2), that

$$\begin{aligned}h^{-1}(G(h)-1)\int_0^t G(s)a\,ds&=h^{-1}\int_0^t(G(s+h)-G(s))a\,ds\\&=h^{-1}(\int_h^{t+h}G(\sigma)ad-\int_0^t G(\sigma)ad\sigma)=h^{-1}(\int_t^{t+h}G(\sigma)ad\sigma-\int_0^h G(\sigma)ad\sigma)\\&=h^{-1}\int_0^h G(s)(G(t)-1)a\,ds\to(G(t)-1)a\quad\text{in}\quad A\end{aligned}$$

as $h \to +0$. Thus $\int_0^t G(s)a\,ds \in D(\Lambda)$. But

$$t^{-1}\int_0^t G(s)a\,ds \to a \quad \text{in} \quad A \quad \text{as} \quad t \to +0,$$

and the density follows.

It remains to prove that Λ is closed. Assume therefore that $a_n \to a$ in A and $\Lambda a_n \to b$ in A as $n \to +\infty$. Then we have

$$G(t)a_n - a_n = \int_0^t G(s)\Lambda a_n ds \to \int_0^t G(s)b\,ds \quad \text{in} \quad A \quad (n \to +\infty).$$

It follows that

$$G(t)a - a = \int_0^t G(s)b\,ds,$$

and thus $a \in D(\Lambda)$ and $\Lambda a = b$. Consequently, Λ is closed and, with the graph norm, $D(\Lambda)$ is a Banach space. □

We shall now give a characterization of the interpolation space $(A, D(\Lambda))_{\theta,p}$. This is the main result of this section.

6.7.3. Theorem. *Let $\{G(t)\}$ be an equi-bounded, strongly continuous semi-group of operators on A, with infinitesimal generator Λ. Then we have*

(3) $$K(t, a; A, D(\Lambda)) \sim \omega(t, a) + \min(1, t)\,\|a\|_A \quad (a \in A),$$

where

$$\omega(t, a) = \sup_{s<t} \|G(s)a - a\|_A,$$

and also

(4) $$\|a\|_{(A, D(\Lambda))_{\theta,p}} \sim \|a\|_A + \left(\int_0^\infty (t^{-\theta}\omega(t, a))^p dt/t\right)^{1/p} \quad (0<\theta<1, 1 \leqslant p \leqslant \infty).$$

If A is reflexive we have

(5) $$\|a\|_{D(\Lambda)} \sim \sup_t t^{-1} K(t, a; A, D(\Lambda)),$$

or, equivalently, $(A, D(\Lambda))_{1,\infty} = D(\Lambda)$.

Proof: (4) is clearly a consequence of (3). To prove (3) let $a = a_0 + a_1$, where $a_0 \in A$ and $a_1 \in D(\Lambda)$. Then

$$\omega(t, a) = \sup_{s<t} \|G(s)a - a\|_A \leqslant \sup_{s<t} \|G(s)a_0 - a_0\|_A + \sup_{s<t} \|G(s)a_1 - a_1\|_A \leqslant (M+1)\,\|a_0\|_A + Mt\,\|\Lambda a_1\|_A,$$

by (iii) and (2). Noting that $\min(1, t)\,\|a\|_A \leqslant K(t, a)$, we have one half of (3). For the other half of (3), we have to find a suitable decomposition $a = a_0 + a_1$

($a_0 \in A$, $a_1 \in D(\Lambda)$) of a. If $t \geqslant 1$ then $K(t, a) = \|a\|_A$ (see the proof of Theorem 3.4.1) and the inequality follows. If $0 < t < 1$ we put

$$a_1 = t^{-1} \int_0^t G(s) a\, ds,$$
$$a_0 = a - a_1 .$$

Then, as in the proof of Lemma 6.7.2, $a_1 \in D(\Lambda)$, and

$$\begin{aligned} K(t, a) &\leqslant \|a_0\|_A + t\,\|a_1\|_{D(\Lambda)} \\ &= \|t^{-1} \int_0^t (G(s)a - a)\, ds\|_A + \|\int_0^t G(s) a\, ds\|_A + \|\Lambda \int_0^t G(s) a\, ds\|_A \\ &\leqslant \omega(t, a) + M \cdot t\,\|a\|_A + \|G(t)a - a\|_A \leqslant C(\omega(t, a) + t\,\|a\|_A), \end{aligned}$$

which completes the proof of (3).

There remains the proof of (5). Assume therefore that A is reflexive. Then $D(\Lambda)$ is reflexive too, since it may be identified with the closed subspace $\{(a, \Lambda a) \mid a \in D(\Lambda)\}$ of $A \times A$, and $A \times A$ is reflexive. One half of (5) is obvious, since $t^{-1} K(t, a) \leqslant \|a\|_{D(\Lambda)}$.

For the other half of (5), we choose a, with $\sup_t t^{-1} K(t, a)$ finite, and a decomposition $a = a_0(t) + a_1(t)$ of a, such that

$$\|a_0(t)\|_A + t\,\|a_1(t)\|_{D(\Lambda)} = O(t) \qquad (t \to +0).$$

Then, since $D(\Lambda)$ is reflexive and its closed unit ball is accordingly weakly compact, there is a subsequence $(b_j)_{j=1}^{\infty}$ of the sequence $(a_1(1/n))_{n=1}^{\infty}$, which converges weakly in $D(\Lambda)$ to an element b. But $a_1(t) = a - a_0(t) \to a$ in A $(t \to +0)$, and thus $b_n \to a$ weakly in A $(n \to +\infty)$. Since $D(\Lambda)$ is dense in A, it follows that $a = b$, and thus $a \in D(\Lambda)$. This completes the proof of (5). □

As an application of the previous theorem, consider the translation group, defined by $G(t)a(x) = a(x+t)$ on $L_p(\mathbb{R}, dx)$ $(1 \leqslant p < \infty)$. Then the infinitesimal generator is given by $\Lambda a(x) = a'(x)$ and $\omega(t, a) = \sup_{s<t} \|\Delta_s a\|_p$. From Theorem 6.2.5, we infer that (equivalent norms)

$$(L_p, D(\Lambda))_{\theta,q} = B_{pq}^{\theta} \qquad (0 < \theta < 1, 1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty).$$

An analogous result holds for the general semi-groups we considered in the example at the beginning of this section. This result gives another characterization of the Besov spaces.

6.7.4. Theorem. *Let* $\{G(t)\}$ *and* A *be as in Example* 6.7.1. *Then, with* $s = \theta m$,

$$(A, D(\Lambda))_{\theta,q} = B_{pq}^{s} \qquad (0 < \theta < 1, 1 \leqslant p, q \leqslant \infty).$$

Proof: Let $a \in B^s_{pq}$, and let ψ and φ_k be defined as in Section 6.2. As in the proof of Theorem 6.2.5, we see that

$$\|A\psi * a\|_A \leqslant C\|\psi * a\|_p,$$
$$\|A\varphi_k * a\|_A \leqslant C2^{mk}\|\varphi_k * a\|_p.$$

Thus by Lemma 6.7.2 we obtain

$$\omega(t, a) = \sup_{s<t}\|G(s)a - a\|_A$$
$$\leqslant C(\min(1, t)\|\psi * a\|_p + \textstyle\sum_{k\geqslant 1}\min(1, t2^{mk})\|\varphi_k * a\|_p).$$

For $s = \theta m$ we conclude that

$$\|a\|_{(A, D(A))_{\theta,q}} \leqslant C(\|a\|_A + (\textstyle\int_0^\infty (t^{-\theta}\omega(t, a))^q\, dt/t)^{1/q}) \leqslant C\|a\|^s_{pq},$$

by Theorem 6.7.3 and the fact that $B^s_{pq} \subset H^0_p = L_p$.

Next, we assume that $a \in (A, D(A))_{\theta,q}$. We have the estimates

$$\|\varphi_k * a\|_p = \|\mathscr{F}^{-1}\{\hat{\varphi}_k\cdot(\exp(-H(2^{-k}\cdot)) - 1)^{-1}\mathscr{F}(G(2^{-mk})a - a)\}\|_p$$
$$\leqslant C\omega(2^{-mk}, a) \qquad (k \geqslant 1),$$
$$\|\psi * a\|_p \leqslant C\|a\|_p = C\|a\|_A,$$

since $\varphi(2^{-k}\cdot)(\exp(-H(2^{-k}\cdot)) - 1)^{-1} \in M_p$ $(1 \leqslant p \leqslant \infty)$, with norm independent of k. These estimates imply that

$$\|a\|^s_{pq} \leqslant C(\|a\|_A + (\textstyle\sum_{k=1}^\infty (2^{sk}\omega(2^{-mk}, a))^q)^{1/q}) \leqslant C\|a\|_{(A, D(A))_{\theta,q}},$$

by Theorem 6.7.3. □

6.8. Exercises

1. Give an example to show that the conclusion in the Mihlin multiplier theorem (6.1.6) does not remain valid for $p = 1$.

2. State and prove the analogue of M. Riesz's theorem on conjugate functions with $\mathbb{R}^n$ instead of the torus $\mathbb{T}$. (See Exercise 4 in Chapter 1.) (Use, e.g., the Riesz transforms $\xi_j/|\xi|$ $(j = 1, 2, \ldots, n)$.)

3. Prove that if, for all integers k and all α with $|\alpha| \leqslant L$ for some integer $L > n/2$,

$$|\xi|^{|\alpha|}\|D^\alpha\rho(\xi)\|_{L(H_0, H_1)} \leqslant a_k, \quad \text{for } 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}$$

and if $\sum_{-\infty}^{\infty} a_k < \infty$, then $\rho \in M_1(H_0, H_1)$ and

$$\|\rho\|_{M_1} \leqslant C \sum_{-\infty}^{\infty} a_k .$$

4. (Löfström [2]). Let g be an infinitely differentiable function on $(0, \infty)$, such that for $j = 0, 1, 2, \ldots$

$$|g^{(j)}(u)| \leqslant C_j u^{-j} \min(u^\alpha, u^{-\beta}), \qquad u \in (0, \infty)$$

where $\alpha > 0$, $\beta > 0$. Moreover, let H be an infinitely differentiable and positive function on $\mathbb{R}^n \setminus \{0\}$, which is positively homogeneous of order $m > 0$. Put

$$\rho(\xi) = g(H(\xi))$$

and prove, using the previous exercise, that $\rho \in M_1$.

The following four exercises indicate other possible ways of defining the spaces B^s_{pq} and H^s_p.

5. Prove that $(f \in \mathscr{S}')$

$$\begin{aligned} \|f\|_p + \textstyle\sum_{|\alpha| = N} \|D^\alpha f\|_p &\sim \textstyle\sum_{|\alpha| \leqslant N} \|D^\alpha f\|_p \\ &\sim \|f\|_p + \textstyle\sum_{j=1} \|\partial^N f / \partial x_j^N\|_p \qquad (1 < p < \infty) . \end{aligned}$$

6. (Peetre [32]). Let the sequence $(\psi_\nu)_\nu$ of functions $\psi_\nu \in \mathscr{S}$ be such that

$$\begin{cases} \operatorname{supp} \psi_\nu \subset [-2^{-\nu}, 2^{-\nu}] , \\ |\psi_\nu^{(j)}(x)| \leqslant C_j 2^{\nu(1+j)} \qquad (j = 0, 1, 2, \ldots) , \\ \int_{\mathbb{R}} x^j \psi_\nu(x)\, dx = 0 \qquad (j = 0, 1, \ldots, k-1) , \\ \sum_{\nu=-\infty}^{\infty} \psi_\nu(x) = \delta(x) . \end{cases}$$

Prove that $(f \in \mathscr{S}')$

$$\|f\|\dot{}^{\,s}_{pq} \sim \left(\textstyle\sum_{\nu=-\infty}^{\infty} (2^{\nu s} \|\psi_\nu * f\|_p)^q\right)^{1/q}, \qquad (s \in \mathbb{R}, 0 < p, q \leqslant \infty)$$

where the definition of $\|\cdot\|\dot{}^{\,s}_{pq}$ is the obvious extension to $0 < p, q \leqslant \infty$ of that in 6.3.
Hint: Superpose sequences of the type $(\psi_\nu^{(h)})_\nu$ to get a sequence as in Lemma 6.1.7, and conversely.

The following definition is essentially Besov's [1].

7. Let $0 < s < 1$. Show that

$$B^s_{pq} = \{f \in \mathscr{S}' \mid \|f\|_p + (\textstyle\int_{\mathbb{R}^n} (|h|^{-s} \|\Delta_h f\|_p)^q \, dh / |h|^n)^{1/q} < \infty\} ,$$

where B^s_{pq} is defined in Section 2. State and prove an analogue for arbitrary $s>0$.

Hint: $(A, \bigcap_{\nu=1}^n D(\Lambda_\nu^m))_{\theta,p} = \bigcap_{\nu=1}^n (A, D(\Lambda_\nu^m))_{\theta,p}$, see Section 9.

8. (Taibleson [1]). Assume that $0<s<1$. Let $u=u(x,t)$ be the solution of the initial value problem:

$$\begin{cases} \partial^2 u/\partial t^2 = -\Delta u & (t>0), \\ u=f & (t=0). \end{cases}$$

Prove that

$$B^s_{pq} = \{ f \in \mathscr{S}' \mid f \in L_p, (\int_0^\infty (t^{1-s} \|\partial u/\partial t\|_p)^q dt/t)^{1/q} < \infty \},$$

where B^s_{pq} is to be interpreted as in Section 2 (Cf. Exercise 24.)

9. Prove that

(a) $B^{n/p}_{p1} \subset L_\infty$,

(b) B^s_{p1} is a Banach algebra under pointwise multiplication if $s \geqslant n/p$,

(c) B^s_{pq} is a Banach algebra under pointwise multiplication if $s > n/p$.

What can you prove about H^s_p?

10. Show that if $f \in B^{n/p}_{p1}$ then $(\sum_{x \in \mathbb{Z}^n} |f(x)|^p)^{1/p}$ is finite, where $\mathbb{Z}$ denotes the set of all integers. (Cf. Peetre [32].)

Hint: Interpolate between H^n_1 and L_∞. Use $B^{n/p}_{p1} \subset (H^n_1, L_\infty)_{\theta,p}$.

11. ("Riemann's second theorem": see Zygmund [1]). Consider

$$f(x) = \sum_{n \neq 0} a_n e^{inx},$$

where $a_n = O(1)$. Put

$$F(x) = \sum_{n \neq 0} a_n n^{-2} e^{inx}.$$

Show that $F \in B^1_{\infty\infty}$. Generalize to the case where

$$F(x) = \sum_{n \neq 0} a_n n^{-\alpha} e^{inx}.$$

Hint: Write $F = F_0 + F_1$ with $F_0 = \sum_{|n| \geqslant t} a_n n^{-2} e^{in\cdot}$.

12. (a) Let $f \in \mathscr{S}'$ and assume that $\|f\|^s_{pq} < \infty$ (see 6.3). Show that

$$D^\alpha f = \sum_{k=-\infty}^{\infty} D^\alpha \varphi_k * f \quad (\text{in } \mathscr{S}')$$

for all α with $|\alpha|>s-n/p$. (This obviously means that $f=\sum_{k=-\infty}^{\infty}\varphi_k*f$ (in $\mathscr{S}'$) if f is taken modulo polynomials of degree at most $[s-n/p]$.) Moreover, if $q=1$, show that

$$D^\alpha f=\sum_{k=-\infty}^{\infty} D^\alpha\varphi_k*f \qquad (\text{in } \mathscr{S}')$$

for all α with $|\alpha|\geqslant s-n/p$.

(b) State and prove the analoguous result for $\|\cdot\dot{\|}_p^s$.

13. Verify that $\sum_{-\infty}^{\infty}\varphi_k*f$ converges in $\dot{H}_p^{s_0}+\dot{H}_p^{s_1}$ if $f\in\dot{B}_{pq}^s$, $s_0<s<s_1$. (This is a part of the proof of Theorem 6.3.1.)

14. Show that spaces $\dot{H}_p^s$ and $\dot{B}_{pq}^s$ are not, in contrast to H_p^s and B_{pq}^s, monotone in s.

15. Show that

$$B_{pq}^s=L_p+\dot{B}_{pq}^s \qquad (s<0,\ 1\leqslant p,q\leqslant\infty),$$
$$H_p^s=L_p+\dot{H}_p^s \qquad (s<0,\ 1\leqslant p\leqslant\infty).$$

Hint: Cf. the proof of Theorem 6.3.2.

There is a classical result by Bernstein [2] concerning absolutely convergent Fourier series. (See Zygmund [1].) The following three exercises extend Bernstein's result. (For applications, see Peetre [4] (summability of Fourier integrals) and Löfström [2] (approximation).)

16. Prove that

$$\mathscr{F}: \dot{B}_{21}^{n/2}\subset L_1.$$

Verify that this implies the inclusion

$$\dot{B}_{21}^{n/2}\subset M_1.$$

17. Show that

$$(M_{p_0}, M_{p_1})_{\theta,1}\subset M_p,$$

where $1/p=(1-\theta)/p_0+\theta p_1$, $1\leqslant p_0, p_1\leqslant\infty$ and $0<\theta<1$. Deduce from this the inclusion

$$(M_{p_0}, M_{p_1})_{\theta,q}\subset M_p,$$

where $1/q>1/2-1/p$ and $2<q<\infty$.

18. Peetre [4]. Prove that if $\rho\in\dot{B}_{q1}^{n/q}$ then $\rho\in M_p$, provided that $1/q>1/p-1/2$ and $2<q<\infty$.

Hint: Use Exercise 16 and 17.

19. Show that if ρ satisfies the conditions in the Mihlin multiplier theorem (6.1.6) then

$$\mathscr{F}^{-1}\rho *: \dot{B}^s_{pq} \to \dot{B}^s_{pq} \quad (s \in R, 1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty).$$

20. (Brenner-Thomée-Wahlbin [1]). Let G be a given function with $\hat{G}$ being infinitely differentiable with compact support on $[-1,1]$ and $\hat{G}(0)=1$. Put

$$G_{a,b}(x) = \sum_{j=1}^{\infty} \exp(i2^j x) 2^{-aj} j^{-b} G(x),$$
$$H_{a,b}(x) = |x|^a (\log(1/|x|))^{-b} \hat{G}(x).$$

Find necessary and sufficient conditions for $G_{a,b} \in B^s_{pq}$ and $H_{a,b} \in B^s_{pq}$ respectively.

21. Show that the inclusions (Theorem 6.2.4)

$$B^s_{p1} \subset H^s_p \subset B^s_{p\infty} \quad (s \in \mathbb{R}, 1 \leqslant p \leqslant \infty)$$

are the best possible in their dependence on the second lower index. (This means that 1 and ∞ cannot be replaced with a q, $q>1$ and $q<\infty$ respectively.)

Hint: Use Exercise 20.

22. Show that the inclusions (Theorem 6.4.4)

$$B^s_{pp} \subset H^s_p \subset B^s_{p2} \quad (s \in \mathbb{R}, 1<p \leqslant 2),$$
$$B^s_{p2} \subset H^s_p \subset B^s_{pp} \quad (s \in \mathbb{R}, 2 \leqslant p < \infty)$$

cannot be improved, i.e. the second lower index p and 2 in the first line cannot be replaced with a q, $q>p$ and $q<2$ respectively.

Hint: See Exercise 20.

23. Show that the inclusions (Theorem 4.7.1)

$$\bar{A}_{\theta,1} \subset \bar{A}_{[\theta]} \subset \bar{A}_{\theta,\infty} \quad (0<\theta<1),$$

where $\bar{A}$ is a compatible Banach couple, cannot be improved, i.e. 1 and ∞ cannot, in general, be replaced with a q, $q>1$ and $q<\infty$ respectively.

Hint: Use Exercise 22.

24. (Taibleson [1]). Put

$$(G(t)f)(x) = \pi^{-(n+1)/2}\Gamma((n+1)/2)\int_{\mathbb{R}^n} t(t^2+|y|^2)^{-(n+1)/2} f(x-y)\,dy$$

and define the space $\Lambda(\alpha; p, q)$ for $\alpha > 0$ by means of the norm

$$\|f\|_{\alpha;p,q} = (\int_0^\infty (t^{\bar{\alpha}-\alpha} \|\partial^{\bar{\alpha}} G(t) f/\partial t^{\bar{\alpha}}\|_p)^q dt/t)^{1/q},$$

where $\bar{\alpha}$ is the smallest integer $> \alpha$. Show that

$$\Lambda(\alpha; p, q) = B^\alpha_{pq} \quad \textit{(equivalent norms)}.$$

25. Let $G(t)$ be defined as in the previous exercise and let χ be a given function such that $\chi \in C_0^\infty([-1, 1])$, $\chi(0) = 1$. Put

$$(L_+ f)(t, x) = \chi(t)(G(t) f)(x), \qquad 0 \leqslant t < \infty.$$

Prove that

$$\|D_t^\mu D_x^\alpha (L_+ f)(t, \cdot)\|_p \leqslant C \min(t^{-m} \|f\|_p, \|f\|^m_{p1}),$$

if $\mu + |\alpha| = m$ and $1 < p < \infty$. Deduce that

$$\|L_+ f\|_{H^m_p(\mathbb{R}^{n+1}_+)} \leqslant C \|f\|_{B^{m-1/p}_{pp}(\mathbb{R}^n)}, \qquad m = 1, 2, \ldots .$$

26. Put

$$(Fg)(t, x) = \begin{cases} g(t, x) & \text{if } t \geqslant 0, \quad x \in \mathbb{R}^n, \\ \sum_{j=1}^N a_j g(-jt, x), & \text{if } t < 0, \quad x \in \mathbb{R}^n, \end{cases}$$

and choose $a_1, \ldots, a_N$ so that

$$F: H^m_p(\mathbb{R}^{n+1}_+) \to H^m_p(\mathbb{R}^{n+1})$$

for $m = 0, 1, \ldots, N$. Then put $L = F L_+$, where L_+ is as in the previous exercise. Show that $\operatorname{Tr} Lf = f$ and

$$L: B^{s-1/p}_{pp}(\mathbb{R}^n) \to H^s_p(\mathbb{R}^{n+1})$$
$$L: B^{s-1/p}_{pq}(\mathbb{R}^n) \to B^s_{pq}(\mathbb{R}^{n+1}),$$

for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $s > 1$.

27. (a) Let G be a strongly continuous, equi-bounded semi-group on a Banach space A and let Λ be the infinitesimal generator of G. Let A^m be the domain of Λ^m and, for $m = 1, 2, \ldots$, put

$$\rho_m(t, a) = t^{1-m} \sup_{0 < s \leqslant t} \|G(s)a - \sum_{n=0}^{m-1} t^n \Lambda^n a/n!\|.$$

Prove that there is a positive constant $C_m > 0$ such that

$$C_m^{-1} K(t, a; A^{m-1}, A^m) \leqslant \rho_m(t, a) + \min(1, t) \|a\|_{A^{m-1}}$$
$$\leqslant C_m K(t, a; A^{m-1}, A^m).$$

(b) (Peetre [10]). Put

$$\omega_m(t, a) = \sup_{0<s\leqslant t} \|(G(s)-1)^m f\|,$$

and prove that there is a positive constant C_m such that

$$C_m^{-1} K(t, a; A, A^m) \leqslant \omega_m(t, a) + \min(1, t)\|a\|_A \leqslant C_m K(t, a; A, A^m).$$

28. (a) (Butzer-Berens [1]). An equi-bounded, strongly continuous semi-group G with infinitesimal generator Λ is said to be *holomorphic* if $G(t)a \in D(\Lambda)$ for all a and

$$\|\Lambda G(t)a\| \leqslant Ct^{-1}\|a\|, \quad t>0.$$

Prove that the space $(A, A^m)_{\theta,q}$, $0<\theta<1$, is given by the condition

$$\left(\int_0^\infty (t^{m-\alpha}\|\Lambda^m G(t)a\|)^q dt/t\right)^{1/q} < \infty$$

where $0<\alpha<m$, $\alpha/m=\theta$.

(b) Prove that the semi-groups on $L_p(\mathbb{R}^n)$ discussed in Section 6.7 are holomorphic.

29. (Triebel [3]). Let F^s_{pq} be the space of all $f \in \mathscr{S}'$ such that f has the representation $f=\sum_{j=0}^\infty f_j$ where $(f_j)_0^\infty \in L_p(l^s_q)$ and $\operatorname{supp}\hat{f}_j \subset \{\xi: 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$ for $j=1,2,\ldots$, $\operatorname{supp}\hat{f}_0 \subset \{\xi: |\xi|\leqslant 2\}$. The norm on F^s_{pq} is

$$\|f\|_{F^s_{pq}} = \inf_{f=\sum f_j} \|(f_j)_0^\infty\|_{L_p(l^s_q)}.$$

Prove that

$$\|f\|_{F^s_{pq}} \sim \|(\varphi_j * f)_0^\infty\|_{L_p(l^s_q)}, \quad (\varphi_0=\psi).$$

Moreover, show that $(F^s_{pq})' = F^{-s}_{p'q'}$ if $1<p, q<\infty$ and that $F^s_{p2}=H^s_p$. Prove also that

$$(F^{s_0}_{pq_0}, F^{s_1}_{pq_1})_{\theta,q} = B^s_{pq},$$
$$(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p} = B^s_{pp},$$
$$(F^s_{p_0q}, F^s_{p_1q})_{\theta,p} = F^s_{pq},$$
$$(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{[\theta]} = F^s_{pq},$$

for suitable values of the parameters.

30. Let $P=P(D)$ be an elliptic partial differential operator of order m with constant coefficients in $\mathbb{R}^n$, such that $P(t\xi)=t^mP(\xi)$, $t>0$, $\xi\in\mathbb{R}^n$. Show that $(f\in\mathscr{S}')$

$$\|f\|_p^{s+m} \sim \|P(D)f\|_p^s \quad (s\in\mathbb{R}, 1<p<\infty),$$
$$\|f\|_{pq}^{s+m} \sim \|P(D)f\|_{pq}^s \quad (s\in\mathbb{R}, 1\leqslant p\leqslant\infty, 1\leqslant q\leqslant\infty).$$

(Cf. Hörmander [2], for example.)

31. Show that the following inequality of Gagliardo-Nirenberg type holds: ($f \in \mathscr{S}'$)

$$\|I^s f\|_p \leqslant C \|I^{s_0} f\|_{p_0}^{1-\theta} \|I^{s_1} f\|_{p_1}^{\theta} \qquad (s_0, s_1 \in \mathbb{R},\ 1 < p_0, p_1 < \infty,\ 0 < \theta < 1)$$

where

$$s = (1-\theta)s_0 + \theta s_1 \text{ and } (1-\theta)/p_0 + \theta/p_1 = 1/p.$$

In Exercise 1.6.14 we defined the space $H_p(\mathbb{T})$, by means of the norm

$$\|f\|_{H_p(\mathbb{T})} = \sup_{r<1} (\int_{\mathbb{T}} |f(re^{i\theta})|^p \, d\theta)^{1/p}.$$

Similarly, $H_p(\mathbb{R})$ *is the space of all functions holomorphic in the upper half-plane* $\operatorname{Im} z > 0$ *such that*

$$\|f\|_{H_p(\mathbb{R})} = \sup_{y>0} (\int_{\mathbb{R}} |f(x+iy)|^p \, dx)^{1/p} < \infty.$$

There are of course n-dimensional versions of these spaces. The following three exercises are meant to point out some recent extensions of classical results to the case $n \geqslant 1$.

32. (Fefferman-Rivière-Sagher [1]). Let T be a linear operator, such that

$$T: (H_{p_0}, H_{p_1}) \to (L_{q_0}, L_{q_1}).$$

Prove that

$$T: H_p \to L_q$$

if

$$1/p = (1-\theta)/p_0 + \theta/p_1, \qquad 1/q = (1-\theta)/q_0 + \theta/q_1,$$
$$0 < \theta < 1, \qquad 0 < p_0, p_1 \leqslant \infty, \qquad 1 \leqslant q_0, q_1 \leqslant \infty,\ p \leqslant q.$$

Hint: Use Formula (2) in Section 9.

33. (Peetre [28]). Prove that if $f \in H_p$ $(0 < p \leqslant 2)$ then

$$(\int_{\mathbb{R}^n} |\mathscr{F} f(\xi)|^p |\xi|^{-n(2-p)} d\xi)^{1/p} < \infty.$$

Hint: Use the fact that $f \in H_p$ $(0 < p < 1)$ implies $|\mathscr{F} f(\xi)| \leqslant C|\xi|^{n(1/p-1)}$ and 1.4.1.

34. (Peetre [28]). Prove that $(\varepsilon > 0,\ 0 < p < 1)$

$$I^{\varepsilon}: H_p \to H_q$$

provided that $1/q = 1/p - \varepsilon/n > 0$, $p > 0$.

Hint: Show that $I^{\varepsilon}: H_p \to \dot{B}^s_{q\infty}$ if $1/q = 1/p + (s-\varepsilon)/n$, and that $\dot{B}^0_{qp} \subset H_q$.

35. (Mitjagin-Semenov, personal communication). Consider the spaces C^j, consisting of j times continuously differentiable real-valued functions on the interval $[-1,1]$. The (semi-) norm is given by $\|f\|_j=\sup_x|D^j f(x)|$ $(f\in C^j)$.

Define the family of operators T_ε $(0<\varepsilon\leqslant 1)$ by the formula

$$T_\varepsilon f(x)=\int_{-1}^{1}\frac{x}{x^2+y^2+\varepsilon^2}(f(y)-f(0))\,dy\,.$$

Use these operators to show that C^1 is not an interpolation space with respect to the couple (C^0, C^2). (Cf. Section 6.9 for additional results.)

Hint: Show that $T_\varepsilon\colon(C^0, C^2)\to(C^0, C^2)$ with norms independent of ε, but that $(T_\varepsilon f_\varepsilon)'(0)\to+\infty$ as $\varepsilon\to+0$, where $f_\varepsilon(x)=(x^2+\varepsilon^2)^{1/2}$, although $\|f_\varepsilon\|_j\leqslant C$, $j=0,2$, for all ε $(0<\varepsilon\leqslant 1)$.

6.9. Notes and Comment

The study of Sobolev and Besov spaces has its roots in questions concerning the regularity of solutions of elliptic partial differential equations. Many of the results in this chapter are due to Hardy and Littlewood in the case $n=1$, see Hardy-Littlewood-Pólya [1]. Another early result is the embedding theorem by Sobolev [1] discussed in 6.5. These works were published before 1940. Sobolev [1] defined spaces W_p^N involving weak derivatives of integral order. There are several possibilities of extending the definition to cover the case of fractional derivatives too. Besov [1] used moduli of continuity (see also Nikolski [1]—a survey article), and Taibleson [1] the solution to an initial value problem for the definition of B_{pq}^s. These definitions are essentially equivalent to the one given in Section 2 (see the exercises and Theorem 6.2.5). The spaces H_p^s $(p\neq 2)$ were introduced by several authors around 1960. For $p=2$ they are much older. In contrast to the spaces B_{pq}^s $(p\neq 2)$, the spaces H_p^s are equivalent to Sobolev's spaces W_p^s for non-negative integral values of s and for $1<p<\infty$ (Theorem 6.2.3 and the exercises). Other ways of defining B_{pq}^s are, e.g., via interpolation:

$$B_{pq}^s=(H_{p_0}^{s_0}, H_{p_1}^{s_1})_{\theta,q}$$

as in Theorem 6.2.4, or via approximation as in Theorem 7.4.2. The latter approach is found in Nikolski [1] and is based on the Jackson and Bernstein inequalities. As we have already remarked, the present definition of the spaces B_{pq}^s in Section 2 is due to Peetre [5].Cf. Nikolski-Lions-Lizorkin [1].

Applications of the results of this and the previous chapter to analysis, e.g., partial differential equations and harmonic analysis, can be found in, e.g., Magenes [1] and in Peetre [2, 4, 7, 8], which also contain many references. Cf. 6.8 and 7.6 for additional references.

We remark here that there is a collection of twelve *open problems* under the heading "Problems in interpolation of operations and applications I—II" in Notices Amer. Math. Soc. **22**, 124—126 (1975); ibid. 199—200.

We note that the results in this chapter also hold, mutatis mutandis, when $\mathbb{R}^n$ is replaced by $\mathbb{T}^n$, the n-dimensional torus. (Cf. 7.5.)

Artola [1] proves the following interpolation result after Lions [1]: Let $\bar{A}$ be a compatible Banach couple with $\Delta(\bar{A})$ dense in both A_0 and A_1. *Assume that Mihlin's multiplier theorem holds in* $L_{p_i}(A_i)$ $(1<p_i, i=0,1)$. *If* $u\in L_{p_0}(A_0)$ *and* $D^m u\in L_{p_1}(A_1)$ *then* $D^j u\in L_p(\bar{A}_{[\theta]})$, *where* $\theta=j/m$, $1/p=(1-\theta)/p_0+\theta p_1$ *and* $0<\theta<1$. We do not know whether there is a version with the real interpolation method.

Zafran [1] has given an example which shows that the answer is 'no' to the following question posed by E. M. Stein: *Let* $\rho\in S'$ *be such that*

$$\mathscr{F}^{-1}\rho^*: L_p\to L_{p\infty}.$$

Does it follow that $\rho\in M_p$? This negative answer is evidently important for a consideration of the relation between the Riesz-Thorin and the Marcinkiewicz theorems, or between the complex and the real methods.

Interpolation of the Hardy spaces H_p, it may be argued, deserves a chapter of its own. However, we think that a summary of the main results, with references, should be sufficient. Our reason for this is that the interpolation techniques which have been used are displayed in the previous chapters, in particular in 1.6.

The classical approach to H_p-spaces is via complex function theory (cf. Duren [1] and Zygmund [1] for the one-dimensional case, and Stein [2] for the n-dimensional case). This approach was complemented in 1972 by a real variable characterization $(0<p<1)$ introduced by Fefferman-Stein [1]. Another important result of Fefferman-Stein [1] is that the dual space of H_1 is the space *BMO*, consisting of functions of "bounded mean oscillation".

The first results concerning interpolation of H_p-spaces were obtained by Thorin [2] in 1948 and by Salem-Zygmund [1]. (Cf. 1.6 and 1.7.) Using the results of Fefferman-Stein [1], it is possible to obtain the following theorems:

(1) $$(H_1, L_\infty)_{[\theta]}=H_p \qquad (1/p=1-\theta, 0<\theta<1),$$

(2) $$(H_{p_0}, H_{p_1})_{\theta,p}=H_p \qquad (1/p=(1-\theta)/p_0+\theta/p_1, 0<p_0, p_1<\infty, 0<\theta<1)$$

where the functions in H_p depend on $n\,(\geqslant 1)$ variables. (1) is found in Fefferman-Stein [1] and (2) in Fefferman-Rivière-Sagher [1]. Some applications are given as exercises. A general reference, for the results about H_p-spaces mentioned above, and containing many references, is Peetre [28].

The spaces B^s_{pq} *with* $0<p<1$ have been discussed by Peetre [32]. (Cf. 7.6 for details.) These spaces appear also in Exercise 35 concerned with H_p-spaces. Connected with this is the following result by Peetre [31]: *Consider the real line* $\mathbb{R}$, *and let* $\rho\in S'$. *Assume that*

$$\rho^*: L_p\to L_p$$

for some p, $0<p<1$. *Then* ρ *is a discrete measure of the form*

$$\rho=\sum c_\alpha \delta(x-x_\alpha),$$

where δ *is the Dirac measure,* (x_α) *is an at most countably infinite family of distinct points in* $\mathbb{R}$ *and* $\sum|c_\alpha|^p$ *is finite.*

If, instead of L_p, the Lorentz spaces L_{pq} are used in the definition of B^s_{pq}, the spaces are denoted by $B^{s,r}_{p,q}$, i.e.

$$B^{s,r}_{p,q}=\{f\in S' \mid \|\psi * f\|_{L_{pq}}+(\textstyle\sum_{k=1}^\infty (2^{ks}\|\varphi_k * f\|_{L_{pq}})^r)^{1/r}<\infty\}$$
$$(s\in\mathbb{R},\ 1\leqslant p,q,r\leqslant\infty).$$

Inclusion and interpolation theorems for these spaces are found in Peetre [9] for example.

An attempt to unify the theory of some different spaces of functions subject to certain growth restrictions is the construction of the spaces $L_{p,\lambda}$,

$$L_{p,\lambda}=\{f \text{ measurable in open } \Omega\subset R^n \mid \sup_{x,r}\inf_\tau(\textstyle\int_{S_{x,r}\cap\Omega}|f(x)-\tau|^p dx)^{1/p}\leqslant Cr^\lambda\},$$

going back to Morrey [1]. Interpolation of (a generalization of) these spaces have been treated by Spanne [1] (see also Peetre [12]). In particular, Spanne [1] treats simultaneously interpolation of L_p-, C^α- and $L_{p,\lambda}$-spaces. (Cf. also Miranda [1], who applies complex methods, and Brudnyi [1].)

Another notion has been proposed by Peetre [5] and investigated by Triebel [3], viz. spaces of Lizorkin type:

$$\dot{F}^s_{pq}=\{f\in S' \mid \|\,\|(\varphi_k * f)\|_{l_q}\|_{L_p}<\infty\}.$$

Note that

$$\dot{B}^s_{pq}=\{f\in S' \mid \|(\|\varphi_k * f\|_{L_p})\|_{l_q}<\infty\}.$$

(Cf. also Peetre [30], who treats the case $0<p,q\leqslant\infty$ and proves a multiplier theorem analogous to Mihlin's. Another definition is found in Exercise 29.)

Quite recently, B.S. Mitjagin has personally communicated a result, which he found together with E.M. Semenov, viz. C^1 is not an interpolation space with respect to the couple (C^0, C^2) (cf. Exercise 35). He also states that they have proved that to any pair of integers k, n with $0<k<n$, there exists an operator T, such that $(C^j=C^j(S^1))$

$$T: C^j\to C^j \quad (0\leqslant j\leqslant n,\ j\neq k),$$
$$T(C^k)\not\subset C^k.$$

Moreover, the same statement holds with $W^j_1(S^1)$ instead of $C^j(S^1)$.

6.9.1. The outstanding result here is, of course, the Mihlin multiplier theorem. This theorem appeared in 1939 in Marcinkiewicz [2] involving Fourier series for functions on the n-dimensional torus $\mathbb{T}^n$. His assumptions are the analogues of those in Theorem 6.1.6. Calderón-Zygmund [1] give another version: sufficient conditions for the L_p-continuity of convolution with certain singular kernels in $\mathbb{R}^n$. The Calderón-Zygmund theorem is an important result in the theory of partial differential equations. In contrast to Marcinkiewicz, Calderón-Zygmund use real variable methods. (In fact, our proof is essentially that of Calderón-Zygmund, in the version of Hörmander [1].) The real variable approach is based on their covering lemma (6.1.8). Mihlin [1] extends Marcinkiewicz' result from $\mathbb{T}^n$ to $\mathbb{R}^n$. Hörmander [1] then presents a theorem containing both the Mihlin and the Calderón-Zygmund results. In particular, he makes the assumption expressed by Formula (19), and his proof is founded on the ideas in Calderón-Zygmund [1]. Hörmander also treats the applications to partial differential equations. Several extensions to the vector-valued case have been made (see, e.g., Triebel [3]).

Let us point to the fundamental role of covering lemmas in the present proof of the Mihlin multiplier theorem. (Cf. also Stein [2] for a discussion of covering lemmas in this context.) Firstly, we invoked the Calderón-Zygmund covering lemma. Secondly, in the proof of this lemma, we referred to the Lebesgue differentiation theorem. For the proof of the latter theorem, another covering lemma is used, e.g., F. Riesz's "sunrise in the mountains" lemma (cf. Riesz-Nagy [1]). However, Riesz's lemma is a special case of the Calderón-Zygmund lemma.

Theorem 6.1.3 is found in Hörmander [1].

As a general reference for this section, we mention Stein [2]. (See Section 1.7 for additional references.)

6.9.2. The idea of using a sequence $(\varphi_k)_{-\infty}^{\infty}$ for the definition of the Besov spaces is taken from Peetre [5]. In the case $p=q=2$, this idea has also been used by Hörmander [2], and, in the general case, by Lizorkin, see Nikolski [1]. (Cf. also Shapiro [2].)

Let us also point out here that not all the alternative definitions are valid as such when $1 \leqslant p \leqslant \infty$. This is a consequence of our recourse to the Mihlin multiplier theorem for the proofs of some of those statements. (Cf. the exercises and Nikolski-Lions-Lizorkin [1].)

6.9.3. The presentation of this section caused a dilemma. Either we had to do everything in tiresome detail, using quotient spaces etc. (cf. Exercise 12), or present the results with semi-norms, as they now stand. The drawback of the latter alternative is that we have not treated the real method in the case of semi-normed spaces. However, it is essentially the equivalence theorem that is needed here. A proof of this can be found in Gustavsson [1], and it exhibits no unexpected features. In fact, the homogeneous spaces $\dot{H}_p^s$ and $\dot{B}_{pq}^s$ are constructed mostly for convenience. The philosophy is: Anything that is true for the homogeneous spaces is true, mutatis mutandis, also for the inhomogeneous spaces, and it is technically easier to work in the homogeneous case.

6.9.4. The results given here for the real method are due to Lions-Peetre [1] and to Peetre [5], and those given for the complex method to Calderón [2].

Note that Formula (4) and Formula (7) of Theorem 6.4.5 indicate that the functors C_θ and $K_{\theta,q}$ are not the same (cf. Exercise 22).

6.9.5. In 1938, Sobolev [1] proved a first version of Theorem 6.5.1. See also Schwartz [1].

The inclusions in Theorem 6.5.1 cannot be improved; cf. Exercise 20—23.

6.9.6. As we pointed out in 3.14, the trace theorems were the forerunners of the abstract real interpolation method. These early results are due to Lions [1]. Subsequently, Lions-Peetre [1] developed the real method.

Recently, Jawerth [1] has proved trace theorems in the case $0<p<1$, using direct methods.

6.9.7. For an introduction to the theory of semi-groups, see Butzer-Behrens [1]. The potential to use interpolation, with the real method, rests on Theorem 6.7.3, Formula (3), describing the functional $K(t, a; A, D(\Lambda))$. The interpolation results are due to Lions [1] (see Lions-Peetre [1] and Peetre [10]). In Peetre [10], the couples $(A, D(\Lambda^m))$, $(A, \bigcap_{\nu=1}^n D(\Lambda_\nu^m))$ and $\bigcap_{\nu=1}^n (A, D(\Lambda_\nu^m))$ $(m\geq 1)$ are considered. In particular, if the semi-groups commute,

$$(A, \textstyle\bigcap_{\nu=1}^n D(\Lambda_\nu^m))_{\theta,p} = \bigcap_{\nu=1}^n (A, D(\Lambda_\nu^m))_{\theta,p} \qquad (0<\theta<1,\ 1\leq p\leq \infty).$$

(Cf. Section 3.14 and the references given there.) However, Grisvard [1] proves that

$$(A_0, D(T)\cap A_1)_{\theta,p} = (A_0, D(T))_{\theta,p} \cap (A_0, A_1)_{\theta,p} \qquad (0<\theta<1,\ 1\leq p\leq \infty)$$

where $\bar{A}$ is a Banach couple, $A_1 \subset A_0$ and $D(T)$ is the domain of the closed operator T contained in A_0, under certain restrictions on the resolvent of T and on A_1; in particular A_1 must be invariant. If $A_1 = D(U)$, the domain of another closed operator U, the result holds under conditions on T and U, which do not imply commutativity. (Cf. also Peetre [15, 27].)

Chapter 7

Applications to Approximation Theory

There is a close connection between the classical approximation theory and the theory of interpolation spaces. We indicated this in 1.5. We discuss the link in more detail in the first two sections. In the first section, the main result is that every "approximation space" is a real interpolation space. The theorem makes the K-method (Chapter 3) available in approximation theory. This is then utilized in Section 2 to obtain, i.a., a classical theorem (of Jackson and Bernstein type; see 1.5) concerning the best approximation of functions in $L_p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$ by entire functions of exponential type. In the following sections, 3 and 4, we prove other approximation theorems, using interpolation techniques developed in Chapter 3, 5 and 6. In particular, we treat approximation of operators by operators of finite rank, and approximation of differential operators by difference operators. Additional applications are indicated in Section 7.5 and 7.6, e.g., approximation by spline functions.

7.1. Approximation Spaces

The basic notion of classical approximation theory is the concept of best approximation $E(t, a)$ to a given function a. We now extend this notion to a more general situation.

We consider the category of all quasi-normed Abelian groups (cf. 3.10). Given a couple $\bar{A}=(A_0, A_1)$ and an element $a \in \Sigma(\bar{A})$ we put

$$E(t, a)=E(t, a; \bar{A})=\inf_{\|a_0\|_{A_0} \leqslant t} \|a-a_0\|_{A_1}, \quad 0<t<\infty .$$

The E-functional just defined does not have any norm property. However, it has the following sub-additivity property.

7.1.1. Lemma. *Assume that A_j is c_j-normed. Then $E(t, a)$ is a decreasing function of t and*

$$E(t, a+b) \leqslant c_1(E(\varepsilon t/c_0, a)+E((1+\varepsilon)t/c_0, b))$$

for $0<\varepsilon<1$. Moreover if $E(t, a)=0$ for all $t>0$ then $a=0$.

In order to make the notation less cumbersome, the reader could put $c_0 = c_1 = 1$ in a first reading.

Proof: If $\|a_0\|_{A_0} \leqslant \varepsilon t/c_0$ and $\|b_0\|_{A_0} \leqslant (1-\varepsilon)t/c_0$ then $\|a_0 + b_0\|_{A_0} \leqslant t$. Since

$$\|(a+b)-(a_0+b_0)\|_{A_1} \leqslant c_1(\|a-a_0\|_{A_1} + \|b-b_0\|_{A_1}),$$

we get the first part of the lemma. In order to prove the second part, we note that if $E(t,a)=0$ for all t we can find $a_n \in A_0$, $n=1,2,\dots$ such that $\|a-a_n\|_{A_1} \to 0$ and $\|a_n\|_{A_0} \to 0$ as $n \to \infty$. Thus $a_n \to 0$ in A_0 and $a_n \to a$ in A_1. But A_0 and A_1 are compatible which implies that there is a Hausdorff topological vector space $\mathscr{A}$ containing A_0 and A_1. Thus $a_n \to 0$ and $a_n \to a$ in $\mathscr{A}$ and hence $a=0$. □

We now investigate the connection between the K- and E-functionals. We shall use the concept of the Gagliardo set Γ, defined as follows. Let $\bar{A}$ be a given couple of quasi-normed Abelian groups. To every element $a \in \Sigma(\bar{A})$ we associate a plane set, the *Gagliardo set* of a, defined as the set of all vectors $x=(x_0, x_1) \in \mathbb{R}^2$ such that $a = a_0 + a_1$, for some $a_0 \in A_0$ and $a_1 \in A_1$, with

$$\|a_0\|_{A_0} \leqslant x_0, \qquad \|a_1\|_{A_1} \leqslant x_1.$$

This set will be denoted by $\Gamma(a)$ or $\Gamma(a;\bar{A})$. In general $\Gamma(a)$ is not convex (cf. 7.1.4 below). If A_j is c_j-normed we have, however, the following sub-additivity property:

$$(x_0/c_0, x_1/c_1) \in \Gamma(a), \qquad (y_0/c_0, y_1/c_1) \in \Gamma(b) \Rightarrow x+y \in \Gamma(a+b).$$

It is also plain that

$$x \in \Gamma(a), \qquad x_j \leqslant y_j \Rightarrow y \in \Gamma(a).$$

In terms of $\Gamma(a)$, we have

$$K_p(t,a) = \inf_{x \in \partial\Gamma(a)} (x_0^p + t x_1^p)^{1/p} = \inf_{x \in \Gamma(a)} (x_0^p + t x_1^p)^{1/p}, \qquad 0 < p \leqslant \infty,$$
$$E(t,a) = \inf_{\substack{x \in \partial\Gamma(a)\\ x_0 \leqslant t}} x_1 = \inf_{\substack{x \in \Gamma(a)\\ x_0 \leqslant t}} x_1,$$

where $\partial\Gamma(a)$ denotes the boundary of $\Gamma(a)$. The second formula means that the intersection of $\Gamma(a)$ with the line $x_0 = t$ is a half-line (closed or open) with end-point $(t, E(t,a))$. To put it in another way, $\partial\Gamma(a)$ is the graph of the function $E(t,a)$ completed, if need be, with vertical lines where $E(t,a)$ is discontinuous. In particular,

$$K(t,a) = \inf_s (s^p + t(E(s,a))^p)^{1/p}, \qquad 0 < p \leqslant \infty, \tag{1}$$

see Figure 4.

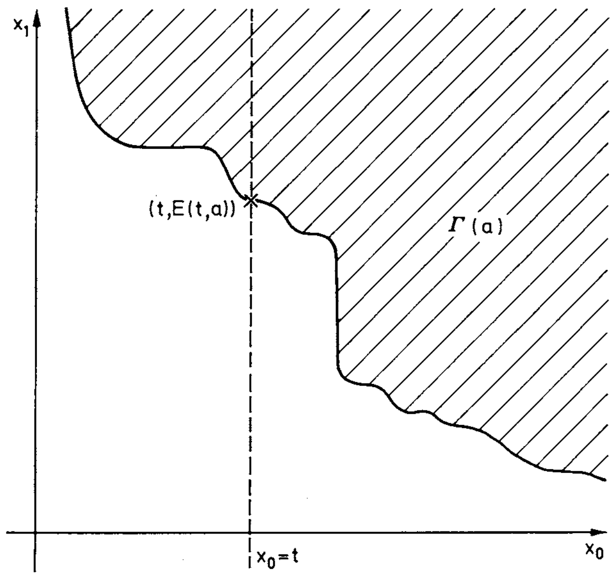


Fig. 4

By Formula (1) we can express the K_p-functional by means of the E-functional. We shall now consider the converse question of expressing the E-functional by means of the K_p-functional in the cases $p=1$ and $p=\infty$.

7.1.2. Lemma. *Given $s>0$, there is a $t>0$ such that*

$$K_\infty(s,a)=t \quad \textit{and} \quad E(t+0,a)\leqslant t/s\leqslant E(t-0,a),$$

where $E(t+0,a)=\limsup_{\tau\to t+0}E(\tau,a)$ and $E(t-0,a)=\liminf_{\tau\to t-0}E(\tau,a)$. In particular, if $E(t)$ is continuous then $K_\infty(t)$ is the inverse of $t/E(t)$.

Proof: Clearly, $\max(x_0, sx_1)=t$ represents, for $x_0\geqslant 0$ and $x_1\geqslant 0$, the finite segments of the lines $x_0=t$ and $x_1=t/s$, see Figure 5. By Formula (1), we have, in addition,

$$K_\infty(s)=s\inf_\tau \max(\tau/s, E(\tau)).$$

These two remarks and an inspection of Figure 5 give the lemma. □

7.1.3. Lemma. *Let $E^*(t,a)$ be defined by the formula*

$$E^*(t,a)=\sup_s s^{-1}(K(s,a)-t).$$

Then $E^*(t,a)$ is the greatest convex minorant of E and

$$E^*(t,a)\leqslant E(t,a)\leqslant(1-\varepsilon)^{-1}E^*(\varepsilon t), \qquad 0<\varepsilon<1.$$

Proof: Writing $\operatorname{ch}\Gamma(a)$ for the convex hull of $\Gamma(a)$, we note that

$$E^*(t,a)=\inf_{\substack{x\in\partial\operatorname{ch}\Gamma(a)\\ x_0\leqslant t}} x_1=\inf_{\substack{x\in\operatorname{ch}\Gamma(a)\\ x_0\leqslant t}} x_1\,,$$

which follows at once from the definition of $E^*(t,a)$ and the above expression for $K_1(t,a)$ in terms of $\Gamma(a)$. (In the expression for $K_1(t,a)$ we may clearly substitute $\operatorname{ch}\Gamma(a)$ for $\Gamma(a)$.) This representation of $E^*(t,a)$ gives the lemma. □

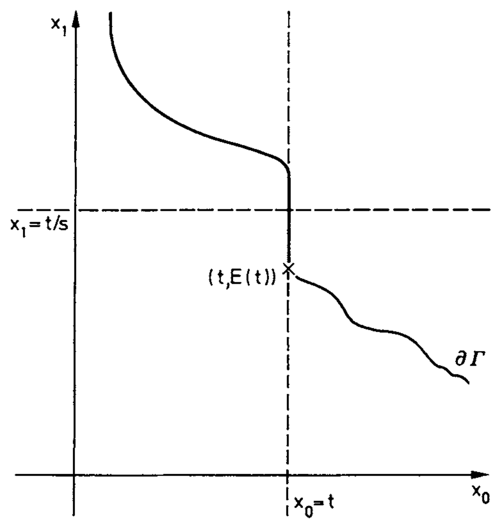


Fig. 5

7.1.4. Corollary. *If $\bar{A}$ is a normed couple*

$$E(t,a)=\sup_s s^{-1}(K_1(s,a)-t)\,.$$

Proof: Clearly, $\Gamma(a)$ is convex if $\bar{A}$ is normed. Then $E^*(t,a)=E(t,a)$ by the above proof. □

We now give the definition of an approximation space.

7.1.5. Definition. *Let $\bar{A}=(A_0,A_1)$ be a given compatible couple of quasi-normed Abelian groups. The approximation space $E_{\alpha q}(\bar{A})$ is the space of all $a\in\Sigma(\bar{A})$ for which*

$$\|a\|_{\alpha q;E}=\Phi_{-\alpha,q}(E(t,a))<\infty\,.$$

Here we take $0<\alpha<\infty$ and $0<q\leqslant\infty$ or $0\leqslant\alpha<\infty$ and $q=\infty$.

7.1.6. Lemma. *Assume that $\bar{A}=(A_0,A_1)$ and that A_j is c_j-normed. Then*

$$a\to\|a\|_{\alpha q;E}$$

defines a c-norm on $E_{\alpha q}(\bar{A})$ *with*

$$c=2c_1\max(c_0^\alpha, c_1^\alpha)\max(1,2^{-1/q'})\max(1,2^{\alpha-1}).$$

Proof: If $\|a\|_{\alpha q;E}=0$ we must have $E(t,a)=0$ for all t and hence $a=0$. By Lemma 7.1.1 we also have

$$\|a+b\|_{\alpha q;E}\leqslant c_1\max(1,2^{-1/q'})(\Phi_{-\alpha}(E(\varepsilon t/c_0, a))+\Phi_{-\alpha,q}(E(1-\varepsilon)t/c_1, b)).$$

Writing $d=2c_1\max(c_0^\alpha, c_1^\alpha)\max(1,2^{-1/q'})$ we see that

$$\|a+b\|_{\alpha q;E}\leqslant d(\varepsilon^{-\alpha}\|a\|_{\alpha q;E}+(1-\varepsilon)^{-\alpha}\|b\|_{\alpha q;E}).$$

Choosing ε so that the two terms on the right hand side are equal we obtain

$$\|a+b\|_{\alpha q;E}\leqslant d(\|a\|_{\alpha q;E}^{1/\alpha}+\|b\|_{\alpha q;E}^{1/\alpha})^\alpha.$$

This gives the result. □

Next, we compare the approximation spaces $E_{\alpha,r}(\bar{A})$ with the interpolation spaces $K_{\theta,q}(\bar{A})$.

7.1.7. Theorem. *Let* $\bar{A}$ *be a quasi-normed couple and put* $\theta=1/(\alpha+1)$, $r=\theta q$. *Then*

$$(E_{\alpha,r}(\bar{A}))^\theta=K_{\theta,q}(\bar{A}).$$

Proof: The norm of a in the space $K_{\theta,q}(\bar{A})$ is equivalent to $\Phi_{\theta,q}(K_\infty(s,a))$ (see 3.11). Let us start with the case $q=\infty$. Then $r=\infty$. Now we choose t according to Lemma 7.1.2. Then we get

$$(t^\alpha E(t,a))^\theta\leqslant s^{-\theta}K_\infty(s,a),$$

which gives $K_{\theta\infty}(\bar{A})\subset(E_{\alpha\infty}(\bar{A}))^\theta$. The converse inclusion is equally obvious, since we have

$$s^{-\theta}K_\infty(s,a)\leqslant(t^\alpha E(t-0,a))^\theta\leqslant\|a\|_{\alpha,\infty;E}^\theta.$$

In the case $q<\infty$ (and $r<\infty$), we integrate by parts and change variables, writing $s=t/E(t,a)$. Note that $s^\theta(K_\infty(s,a)\to 0$ as $s\to 0$ or $s\to\infty$ and that $t^\alpha E(t,a)\to 0$ as $t\to 0$ or $t\to\infty$. Thus

$$\begin{aligned}\int_0^\infty(s^{-\theta}K_\infty(s,a))^q ds/s &\sim -\int_0^\infty K_\infty(s,a)^q\, ds^{-\theta q}=\int_0^\infty s^{-\theta q}dK_\infty(s,a)^q\\ &=\int_0^\infty(t/E(t,a))^{-\theta q}d(t^q)\sim\int_0^\infty(t^\alpha E(t,a))^{\theta q}dt/t\end{aligned}$$

which gives the result. □

As a consequence of the reiteration and power theorems, we obtain

7.1.8. Theorem. *If* $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$ *and* $\alpha_0\neq\alpha_1$ *then*

$$K_{\theta,q}(E_{\alpha_0,r_0}(\bar{A}),E_{\alpha_1,r_1}(\bar{A}))=E_{\alpha,q}(\bar{A}),$$

and

$$(E_{\beta,r}(E_{\alpha_0,r_0}(\bar{A}),E_{\alpha_1,r_1}(\bar{A})))^{\theta}=E_{\alpha,q}(\bar{A}),$$

where $r=\theta q$ *and* $\beta=(\alpha_1-\alpha)/(\alpha-\alpha_0)$ *or, equivalently,* $\theta=1/(\beta+1)$.

7.2. Approximation of Functions

In this section we determine the approximation space $E_{\alpha,q}(\bar{A})$ for certain couples $\bar{A}$. We will get other proofs of some of the results of Chapter 5 and 6, thus giving another interpretation of these results.

Let (U,μ) be a measure space. We recall the definition of the space L_0. The quasi-norm on L_0 is

$$\|f\|_{L_0}=\mu(\operatorname{supp} f),$$

where f is measurable and its support, supp$f$, is any measurable set F such that $f=0$ outside F and $f\neq 0$ almost everywhere on F. We shall now find the value of $E(t,f;L_0,L_\infty)$.

7.2.1. Lemma. *Let* f^* *be the non-increasing rearrangement of* f. *Then*

$$E(t,f;L_0,L_\infty)=f^*(t).$$

Proof: By definition $E(t,f)$ is the infimum of all numbers of the form $\|f-g\|_\infty$, where the μ-measure of the support F of g is at most t. Now put $\tilde{g}(x)=f(x)$ on F and $\tilde{g}(x)=0$ outside F. Then $\|f-\tilde{g}\|_\infty\leqslant\|f-g\|_\infty$. Next, we consider the function

$$g_\sigma(x)=\begin{cases} f(x) & \text{if } |f(x)|>\sigma,\\ 0 & \text{otherwise.}\end{cases}$$

Writing $\tau=\sup\{|f(x)|: x\notin F\}$ we have $\operatorname{supp} g_\tau\subset F$. Thus $\mu(\operatorname{supp} g_\tau)\leqslant t$. Clearly $\|f-g_\tau\|_\infty\leqslant\tau$ and $\|f-\tilde{g}\|_\infty=\tau$. Thus we see that

$$E(t,f)=\inf_\sigma\{\|f-g_\sigma\|_\infty: \mu(\operatorname{supp} g_\sigma)\leqslant t\}.$$

But the right hand side is just the definition of $f^*(t)$. □

From Lemma 7.2.1 and Theorem 7.1.7 we get the following complement of Theorem 5.2.1.

7.2.2. Theorem. *For any $p>0$ and $q>0$ we have*

$$E_{1/p,q}(L_0, L_\infty) = L_{pq} \quad \textit{(equal norms)}.$$

If $\theta = p/(p+1)$ and $r = \theta q$ we therefore have

$$K_{\theta,q}(L_0, L_\infty) = (L_{pr})^\theta .$$

7.2.3. Corollary. *If $\theta = 1/(\beta+1)$, $r = \theta q$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ then*

$$E_{\beta,r}(L_{p_0}, L_{p_1}) = (L_{p,q})^{1/\theta} .$$

The corollary is an immediate consequence of Theorem 7.1.8.

Next, we shall consider approximation spaces between L_p $(1 \leqslant p \leqslant \infty)$ and the space $\mathscr{E}_p$ of all entire functions in L_p of exponential type. Thus $\mathscr{E}_p$ consists of all functions $f \in L_p$ for which $\hat{f}$ has compact support. Let us write

$$\|f\|_{\mathscr{E}} = \sup\{|\xi| : \hat{f}(\xi) \neq 0\} .$$

The space $\mathscr{E}_p$ becomes a quasi-normed (1-normed) vector space if we introduce the functional

$$\|f\|_{\mathscr{E}_p} = \|f\|_{L_p} + \|f\|_{\mathscr{E}} .$$

We shall make use of the following classical inequalities: $(N = 0, 1, 2, \dots)$

(1) $$E(t, f; \mathscr{E}_p, L_p) \leqslant C_N t^{-N} \|f\|_{\dot{H}_p^N}, \quad \text{(Jackson)},$$

(2) $$\|f\|_{\dot{H}_p^N} \leqslant C_N \|f\|_{\mathscr{E}}^N \|f\|_{L_p}, \quad \text{(Bernstein)}.$$

For completeness, we give the proofs here. In order to prove Jackson's inequality, we choose a function $\chi \in \mathscr{S}(R)$, such that

$$\chi(u) = \begin{cases} 1, & u \leqslant \frac{1}{2} \\ 0, & u \geqslant 1 . \end{cases}$$

Put $\hat{\varphi}_t(\xi) = \chi(t^{-N}|\xi|^N)$ and $\hat{\psi}_t = t^N |\xi|^{-N} (\chi(t^{-N}|\xi|^N) - 1)$. Obviously, $\hat{\psi}_t \in M_p$, $1 \leqslant p \leqslant \infty$, by Lemma 6.1.5, with $\|\hat{\psi}_t\|_{M_p}$ independent of t (Theorem 6.1.3). Moreover, $\varphi_t * f \in E_p$ and $\varphi_t * f - f = t^{-N} \psi_t * I^N f$. Thus

$$E(t, f; \mathscr{E}_p, L_p) \leqslant \|\hat{\psi}_t\|_{M_p} t^{-N} \|I^N f\|_{L_p},$$

which gives (1). In order to prove Bernstein's inequality, we put $\hat{\chi}_N(\xi) = \chi(|\xi|/2t)^N$, with χ as in the above proof and $t = \|f\|_{\mathscr{E}}$. Then, arguing as for $\hat{\psi}_t$ above, we have

$$\|I^N f\|_p = \|I^N \chi_N * f\|_p = \|\mathscr{F}^{-1}\{(|\xi|\,\chi(|\xi|/2t))^N \mathscr{F} f\}\|_p \leqslant C t^N \|f\|_p,$$

which proves (2). $\square$

In view of Theorem 7.1.7, (1) is equivalent to

$$t^{-1/(N+1)}K(t,f;\mathscr{E}_p,L_p)\leqslant C_N(\|f\|_{H_p^N})^{1/(N+1)}. \tag{3}$$

Similarly, (2) implies that

$$(\|f\|_{H_p^N})^{1/(N+1)}\leqslant C_N\|f\|_{\mathscr{E}_p}^{1-1/(N+1)}\|f\|_{L_p}^{1/(N+1)}.$$

Thus $(H_p^N)^{1/(N+1)}$ is of class $\mathscr{C}_{1/(N+1)}(\mathscr{E}_p,L_p)$. Using the theorems, the numbers of which stand above the equality signs, we obtain, with $\alpha>0$,

$$\begin{aligned}E_{\alpha,r}(\mathscr{E}_p,L_p)&\overset{(7.1.7)}{=}((\mathscr{E}_p,L_p)_{1/(\alpha+1),r(\alpha+1)})^{\alpha+1}\\&\overset{(3.7.1)}{=}((L_p,(H_p^N)^{1/(N+1)})_{\alpha(N+1)/(\alpha+1)N,r(\alpha+1)})^{\alpha+1}\\&\overset{(3.12.6)}{=}(L_p,H_p^N)_{\alpha/N,r}\\&\overset{(6.4.5)}{=}B_{pr}^{\alpha}.\end{aligned}$$

(Theorem 3.7.1 is valid in the quasi-normed case too, as we remarked in Section 3.11.) Therefore we have proved

7.2.4. Theorem. *For any $\alpha>0$ we have*

$$E_{\alpha,r}(\mathscr{E}_p,L_p)=B_{pr}^{\alpha}.$$

This is a classical result on the best approximation by entire functions. A particular case $(r=\infty,\ 0<\alpha<1)$ is

$$E(t,f;\mathscr{E}_p,L_p)=O(t^{-\alpha}),\qquad t\to\infty,$$

if and only if

$$\omega_p(t,f)=O(t^{\alpha}),\qquad t\to 0.$$

7.3. Approximation of Operators

There is an analogy between the question of finding the approximation space $E_{\alpha,q}(L_0,L_\infty)$ and the question of approximation by means of operators of finite rank. We sketch this analogy briefly.

In this section, we let $\mathfrak{S}_\infty(A, B)$ stand for the space of all bounded linear operators from the Banach space A to the Banach space B. We let $\mathfrak{S}_0(A, B)$ denote the space of all such operators of finite rank. The norms are defined by

$$\|T\|_{\mathfrak{S}_\infty(A,B)} = \sup_{\|a\|_A \leq 1} \|Ta\|_B,$$
$$\|T\|_{\mathfrak{S}_0(A,B)} = \operatorname{rank} T = \dim_B T(A).$$

Now we consider the approximation number

$$E(t, T) = \inf\{\|T - S\|_{\mathfrak{S}_\infty(A,B)} : \operatorname{rank} S \leq t\},$$

and the space $\mathfrak{S}_p(A, B)$ $(0 < p < \infty)$ of all T such that

$$\|T\|_{\mathfrak{S}_p(A,B)} = (\textstyle\int_0^\infty E(t, T)^p dt)^{1/p}.$$

From Theorem 7.1.7 and 7.1.8 we now infer that

$$(\mathfrak{S}_0(A, B), \mathfrak{S}_\infty(A, B))_{\theta, p/\theta} = (\mathfrak{S}_p(A, B))^\theta$$

if $\theta = p/(p+1)$ and thus

$$(\mathfrak{S}_{p_0}(A, B), \mathfrak{S}_{p_1}(A, B))_{\theta, p} = \mathfrak{S}_p(A, B)$$

if $1/p = (1-\theta)/p_0 + \theta/p_1$.

If A and B are Hilbert spaces, then $\mathfrak{S}_p(A, B)$ consists of the p-nuclear operators from A to B. A linear operator T from A to B is *p-nuclear* if it can be represented in the form

$$Ta = \textstyle\sum_{i=1}^\infty \lambda_i < a, a_i' > b_i,$$

where

$$(\textstyle\sum_i |\lambda_i|^p)^{1/p} < \infty, \qquad \|a_i'\|_{A'} \leq 1, \qquad \|b_i\|_B \leq 1.$$

The norm of T is $\inf(\sum_i |\lambda_i|^p)^{1/p}$. (Cf. Exercise 11.)

7.4. Approximation by Difference Operators

In this section we consider the rate of convergence of difference operators.

Let $G(t)$ be the solution operator for the initial value problem

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + P(D)u = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u = f, \quad x \in \mathbb{R}^n, & t = 0. \end{cases}$$

We assume that the differential operator $P(D)$, with D as in 6.1, has constant coefficients and that the polynomial $P(\xi)$ is positive for $\xi \neq 0$ and positively homogeneous of order $m>0$. The solution u of (1) is $u=G(t)f$. Clearly, $G(t)$ is the semi-group of operators considered in the example of Section 6.7. Thus

$$\|u\|_{L_p} \leqslant C\|f\|_{L_p},$$

i.e. (1) is correctly posed.

We shall now approximate to the solution u of (1) by means of a function u_h, which is constructed as the solution of a discrete initial value problem of the form

$$\begin{cases} u_h(x, t+k) = \sum_\alpha e_\alpha u_h(x+\alpha h, t), & t=0, k, 2k, 3k, \dots, \\ u_h(x, 0) = f(x), \end{cases} \tag{2}$$

where $k>0$, and where $\alpha \in \mathbb{R}^n$ are chosen with regard to P. Clearly, u_h depends linearly on f, so we can write $u_h(x, t) = (G_h(t)f)(x)$ where $t=k, 2k, 3k, \dots$. $G_h(k)$ is given by the formula

$$(G_h(k)f)(x) = \sum_\alpha e_\alpha f(x+\alpha h).$$

From (2) we see that

$$u_h(x, t) = (G_h(k)^N f)(x), \quad t=Nk, \quad N=1, 2, \dots.$$

The operator $G_h(t)$ can be characterized by means of the Fourier transform. In fact, we clearly have

$$\mathscr{F}(G_h(k)f)(\xi) = e(h\xi)\hat{f}(\xi),$$

where

$$e(\eta) = \sum_\alpha e_\alpha \exp(i\langle \alpha, \eta\rangle)$$

(the symbol of the difference scheme (2)). Therefore

$$\mathscr{F}(G_h(t)f)(\xi) = e(h\xi)^N \hat{f}(\xi), \quad t=Nk.$$

Assuming that $u_h \to u$ in L_p and using the principle of uniform boundedness, we see that the difference scheme (2) must be stable in the sense that

$$\|u_h\|_{L_p} \leqslant C\|f\|_{L_p}.$$

In terms of Fourier multipliers, this condition could be rephrased as

$$\sup_{N=1,2,\dots} \|e(\xi)^N\|_{M_p} < \infty. \tag{3}$$

Example: Let us consider the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R}, \quad t > 0, \\ u = f, & x \in \mathbb{R}, \quad t = 0. \end{cases}$$

In order to approximate to the solution u, we replace the differential operators by difference operators. For instance, we may replace $\partial u/\partial t$ by $k^{-1}(u_h(x,t+k) - u_h(x,t))$ and $\partial^2 u/\partial x^2$ by $h^{-2}(u_h(x+h,t) - 2u_h(x,t) + u_h(x-h,t))$. Assuming that $kh^{-2} = \lambda$ (a positive constant), we thus replace the continuous initial value problem above by a discrete counterpart

$$\begin{cases} u_h(x,t+k) = u_h(x,t) + \lambda(u_h(x+h,t) - 2u_h(x,t) + u_h(x-h,t)), \\ u_h(x,0) = f(x). \end{cases}$$

In this case we therefore have

$$e(\eta) = 1 + \lambda(e^{i\eta} - 2 + e^{-i\eta}),$$

i.e.

$$e(\eta) = 1 - 4\lambda \sin^2(\eta/2).$$

Here we have stability if $0 < \lambda \leq 1/2$. In fact,

$$\|e(\xi)\|_{M_1} \leq |1 - 2\lambda| + 2\lambda = 1 \quad \text{if} \quad 0 < \lambda \leq 1/2.$$

Thus, by Theorem 6.1.2, if $0 < \lambda \leq 1/2$ then

$$\|e(\xi)^N\|_{M_p} \leq 1. \quad \square$$

We now return to the general case. Our objective is to study the rate of the convergence of $u_h \to u$ as $h \to 0$. More precisely, we want to find the space A_p^σ of all $f \in L_p$ such that $\|G_h(t)f - G(t)f\|_{L_p} \leq Ch^\sigma$, $h \to 0$ (uniformly in $t = k, 2k, \ldots$, $k = \lambda h^m$). Clearly, we shall have to make some assumption on how fast u_h converges to u, when f is a nice function. This can be specified by means of assumptions on the difference $e(h\xi) - \exp(-kP(\xi))$ or, equivalently, by means of assumptions on $P_h(\xi) - P(\xi)$, where

$$P_h(\xi) = -k^{-1} \log e(h\xi), \qquad h|\xi| \quad \text{small}.$$

7.4.1. Definition. *Assume that* $k = \lambda h^m$, *m being the order of* $P(\xi)$. *Then we say that P_h approximates P with degree exactly* $s > 0$, *if*

$$P_h(\xi) - P(\xi) = h^s |\xi|^{m+s} Q(h\xi),$$

where $Q(\eta)$ is infinitely differentiable on $0<|\eta|<\varepsilon_0$ and has bounded derivatives there, and if

$$|Q(\eta)|\geqslant Q_0>0, \quad 0<|\eta|<\varepsilon_0 .$$

In the example considered above, we have

$$P_h(\xi)=-\lambda^{-1}h^{-2}\log(1-4\lambda\sin^2(h\xi/2)),$$

and

$$P(\xi)=\xi^2 .$$

Then it is easily seen that

$$P_h(\xi)-P(\xi)=c_\lambda h^2|\xi|^4+d_\lambda h^4|\xi|^6+O(h^6|\xi|^8), \qquad h|\xi|\to 0,$$

where $c_\lambda=0$ if and only if $\lambda=1/6$ and $d_{1/6}\neq 0$. Thus P_h approximates P of degree exactly 2 if $\lambda\neq 1/6$ and of degree exactly 4 if $\lambda=1/6$.

7.4.2. Theorem. *Assume that $G_h(t)$ is a stable difference operator in the sense of* (3), *and that P_h approximates P with order exactly $s>0$. Then*

$$A_p^\sigma=B_{p\infty}^\sigma, \quad 0<\sigma\leqslant s, \quad 1\leqslant p\leqslant\infty .$$

Moreover, if $f\in L_p$ $(1\leqslant p\leqslant\infty)$ and

$$\lim_{h\to 0}\sup_{t=k,\,2k,\ldots}h^{-s}\|G_h(t)f-G(t)f\|_{L_p}=0,$$

then $f=0$.

Proof: In order to prove $B_{p\infty}^\sigma\subset A_p^\sigma$ we write, with $\varphi_0=\psi$,

$$u_h-u=\textstyle\sum_{j\geqslant 0}(G_h(t)-G(t))\varphi_j*f, \qquad t=Nk .$$

We shall establish the estimates

$$\|\exp(-tP_h)-\exp(-tP)\|_{M_p}\leqslant C$$

and

$$\|(\exp(-tP_h)-\exp(-tP))\textstyle\sum_{l=-1}^{1}\hat\varphi_{j+l}\|_{M_p}$$
$$\leqslant C(\exp(-Dt2^{mj})-\exp(-At2^{mj}))(h2^j)^s, \qquad h2^j\leqslant\varepsilon<\varepsilon_0/2,$$

where $A>D>0$ if ε is small enough, and $\hat\varphi_{-1}\equiv 0$. These two estimates give the desired inclusion, since $\sum_{l=-1}^{1}\varphi_{j+l}*\varphi_j=\varphi_j$,

$$\begin{aligned}h^{-s}\|u_h-u\|_p&\leqslant\textstyle\sum_{j\geqslant 0}h^{-s}\|(G_h(t)-G(t))\varphi_j*f\|_p\\&\leqslant C(\textstyle\sum_{h2^j\leqslant\varepsilon}(\exp(-Dt2^{mj})-\exp(-At2^{mj}))\\&\quad+\textstyle\sum_{h2^j>\varepsilon}(h2^j)^{-s})\|f\|_{p\infty}^s\leqslant C\|f\|_{p\infty}^s .\end{aligned}$$

$$\|u_h-u\|_p\leqslant C\|f\|_{p1}^0 \qquad \text{(Theorem 6.2.4)},$$

and thus, by interpolation,

$$h^{-\sigma}\|u_h-u\|_p\leqslant C\|f\|_{p\infty}^{\sigma}, \qquad 0<\sigma\leqslant s.$$

There remain the two estimates. The first one follows directly from the stability. For the proof of the second one, we write

$$\begin{aligned}&\exp(-tP_h(\xi))-\exp(-tP(\xi))\\&\quad=t|\xi|^m(h|\xi|)^sQ(h\xi)\exp(-tP(\xi))\textstyle\int_0^1\exp(rt|\xi|^m(h|\xi|)^sQ(h\xi))dr.\end{aligned}$$

Invoking Theorem 6.1.3 and Lemma 6.1.5, we obtain

$$\begin{aligned}&\|(\exp(-tP_h)-\exp(-tP))\textstyle\sum_{l=-1}^{1}\hat{\varphi}_{j+l}\|_{M_p}\\&\quad\leqslant Ct2^{mj}(h2^j)^s\exp(-At2^{mj})\textstyle\int_0^1\exp(Brt2^{mj}(h2^j)^s)dr\\&\quad\leqslant C(\exp(-Dt2^{mj})-\exp(-At2^{mj}))(h2^j)^s, \qquad h2^j\leqslant\varepsilon<\varepsilon_0/2,\end{aligned}$$

where $A>0$, $D=A-B\varepsilon^s$, $B>0$. Clearly, $D>0$ if ε is small enough. This is the second estimate.

The converse inclusion $A_p^\sigma\subset B_{p\infty}^\sigma$ is implied by the following estimate, a consequence of Theorem 6.1.3 and Lemma 6.1.5. We have, with $t=Nk=2^{-mj}$, $h=l2^{-j}$, $(j\geqslant 2)$

$$\begin{aligned}\|\varphi_j*f\|_p&\leqslant C\|(\exp(-tP_h)-\exp(-tP))^{-1}\textstyle\sum_{l=-1}^{1}\hat{\varphi}_{j+l}\|_{M_p}\\&\quad\cdot\|G_h(t)f-G(t)f\|_p\\&\leqslant Ch^\sigma\|\exp(tP)(\exp(-th^s|\xi|^{m+s}Q(h\xi))-1)^{-1}\textstyle\sum_{l=-1}^{1}\hat{\varphi}_{j+l}\|_{M_p}\\&\leqslant Ch^\sigma(h2^j)^{-s}, \quad \text{for } N \text{ large enough,}\end{aligned}$$

since

$$|D^\alpha(\exp(-(h2^j)^s|\xi|^{m+s}Q(h2^j\xi))-1)^{-1}|\leqslant C(h2^j)^{-s} \qquad (4^{-1}<|\xi|<4)$$

if $l=h2^j<\varepsilon_0/4$, i.e. N is large enough. Note also that if we know that

$$\sup_t\|G_h(t)f-G(t)f\|_{L_p}=o(h^s)$$

then it follows that $\varphi_j*f=0$ for all j, i.e. that $f=0$. □

7.5. Exercises

1. Let $\hat{k}$ be a given infinitely differentiable function such that $\hat{k}(\xi)=1$ for $|\xi|<1/2$ and $\hat{k}(\xi)=0$ for $|\xi|>1$. Put $k_\lambda(x)=\lambda^n k(\lambda x)$, $x\in\mathbb{R}^n$. Prove that

$$E(\lambda,f;\mathscr{E}_p,L_p)=O(\lambda^{-s}), \qquad \lambda\to\infty,$$

if and only if

$$\|k_\lambda * f - f\|_p = O(\lambda^{-s}), \qquad \lambda \to \infty .$$

2. (Löfström [2]). For $\hat{k} \in M_p$ let k_λ be given by $\mathscr{F}(k_\lambda)(\xi) = \hat{k}(\xi/\lambda)$ $(\lambda > 0)$. Prove that the implication

$$f \in B^s_{p1} \Rightarrow \|k_\lambda * f - f\|_p = O(\lambda^{-s}), \qquad \lambda \to \infty$$

holds if and only if for some $\varepsilon > 0, C > 0$

$$(1) \qquad \| \, |\xi|^{-s}(\hat{k}(\xi) - 1)\varphi(\xi/r)\|_{M_p} \leqslant C, \quad \text{for} \quad 0 < r < \varepsilon .$$

(Here φ is the standard function in the definition of Besov spaces.) Prove also that if (1) holds then we have for $0 < \sigma < s$

$$f \in B^\sigma_{p\infty} \Rightarrow \|k_\lambda * f - f\|_p = O(\lambda^{-\sigma}).$$

3. (Shapiro [1]). For $\hat{\sigma} \in M_\infty$ we define σ_λ as in the previous exercise. Put

$$D_\sigma(t, f) = \sup_{\lambda t \geqslant 1} \|\sigma_\lambda * f\| .$$

Prove that if $\hat{\sigma}(\xi) \neq 0$ on $|\xi| = 1$ and if $\hat{\rho} \in M_\infty$ can be written $\hat{\rho}(\xi) = \hat{\sigma}(\xi)\hat{\tau}(\xi)$ in a neighbourhood of $\xi = 0$, with $\hat{\tau} \in M_\infty$, then there are constants $C > 0$ and $\varepsilon > 0$ such that

$$D_\rho(t, f) \leqslant C(D_\sigma(t, f) + \textstyle\sum_{k=1}^\infty D_\sigma(\varepsilon t 2^{-k}, f)).$$

4. (Shapiro [1]). Let σ be a real, bounded, non-vanishing measure on the real line and assume that $D_\sigma(t, f) = O(t^q)$ as $t \to 0$. Prove that the modulus of continuity $\omega_m(t, f)$ of order m on L_∞ is given by

$$\omega_m(t, f) = \begin{cases} O(t^q) & \text{if} \quad q < m, \\ O(t^q \ln 1/t) & \text{if} \quad q = m, \\ O(t^m) & \text{if} \quad q > m. \end{cases}$$

5. (Löfström [2], Peetre [4]). Put

$$\hat{k}(\xi) = (1 - H(\xi))^\alpha_+, \qquad \xi \in \mathbb{R}^n$$

where H is infinitely differentiable and positive outside the origin and $H(t\xi) = t^m H(\xi)$ for $t > 0$. Prove that $\hat{k} \in M_\infty$ if $\alpha > (n-1)/2$. Prove also that

$$\|k_\lambda * f - f\|_\infty = O(\lambda^{-s}), \qquad (0 < s < m)$$

if and only if $f \in B^s_{\infty\infty}$.

Hint: Use Exercise 18 in Chapter 6 and Exercise 3 above.

6. In this and the next three exercises we conside the n-dimensional torus $\mathbb{T}^n$. Let φ be the standard function in the definition of the Besov spaces and let φ_k be the function whose Fourier coefficients are $\varphi(2^{-k}\xi)$, $\xi \in \mathbb{Z}^n$ ($\mathbb{Z}$ is the set of all integers). Let J_0^s be the operator defined by

$$\mathscr{F}(J_0^s f)(\xi) = (1+|\xi|^2)^{s/2} \hat{f}(\xi), \qquad \xi \in \mathbb{Z}^n,$$

where $\hat{f}(\xi)$ are the Fourier coefficients of f.

(a) Prove that $\|J_0^s \varphi_k * f\|_p \leq C 2^{sk} \|\varphi_k * f\|_p$.

(b) Define $H_p^s(\mathbb{T}^n)$ by means of the norm

$$\|f\|_p^s = \|J_0^s f\|_p.$$

Prove that the norm on

$$(L_p(\mathbb{T}^n), H_p^N(\mathbb{T}^n))_{\theta,r} = B_{pr}^s(\mathbb{T}^n), \qquad s = \theta N,$$

is equivalent to

$$\|f\|_{pr}^s = \|\psi * f\|_p + (\textstyle\sum_{k=1}^\infty (2^{sk} \|\varphi_k * f\|_p)^r)^{1/r}.$$

(c) Define the space $B_{pr}^s(\mathbb{T}^n)$ by means of differentiability conditions as in Theorem 6.2.5.

7. Let $\mathscr{E}^0$ be the space of all trigonometric polynomials on $\mathbb{T}^n$. Let $E_m(f) = E(m, f; \mathscr{E}^0, L_p)$ be the best approximation of f (in L_p-norm) by trigonometric polynomials of degree at most m. Prove, using the method of Section 7.2, the classical result

$$E_m(f) = O(m^{-s}) \quad \text{if and only if} \quad f \in B_{p\infty}^s.$$

Also, find the space of all $f \in L_p(\mathbb{T}^r)$ such that

$$(\textstyle\sum_{n=1}^\infty (n^s E_n(f))^r n^{-1})^{1/r} < \infty.$$

8. (Löfström [2]). Prove by imitating the proof in Exercise 4 of Chapter 6, that if g is infinitely differentiable and satisfies

$$|g^{(j)}(u)| \leq C_j u^{-j} \min(u^\alpha, u^{-\beta}),$$

where $\alpha > 0$, $\beta > 0$, then $g(tH(\xi))$, $\xi \in \mathbb{Z}^n$ are the Fourier coefficients of a function $G_t \in L_1(\mathbb{T}^n)$ and that

$$\|G_t\|_1 \leq C, \qquad 0 < t < \infty.$$

9. (Löfström [2]). Put

$$f_t(x)=(2\pi)^{-n}\sum_{\xi\in\mathbb{Z}^n}\exp(i\langle x,\xi\rangle - tH(\xi))\,\hat{f}(\xi)$$

for $f\in L_p(\mathbb{T}^n)$. Prove that

$$\|f_t-f\|_p=O(t^s) \quad \text{if and only if} \quad f\in B^s_{p\infty}(\mathbb{T}^n).$$

10. Show that

$$E(t,f;L_0,L_p)=(\int_t^\infty (f^*(s))^p ds)^{1/p} \qquad (0<p\leqslant\infty).$$

Deduce from this the inequality in the proof of the Marcinkiewicz theorem given in 1.3.

Hint: Theorem 7.2.2.

11. (Peetre-Sparr [1]). Denote by $\mathcal{N}_p=\mathcal{N}_p(A,B)$ the space of all p-nuclear operators (cf. 7.3) from the Banach space A to the Banach space B. Prove that

$$\mathcal{N}_p\subset\mathcal{S}_q \qquad (0<p<1,\ 1/q>1/p-1),$$

$$\mathcal{S}_p\subset\mathcal{N}_p \qquad (0<p\leqslant\infty).$$

Hint: Use the result in Section 3, and apply Auerbach's lemma: *If* $\operatorname{rank}(T)\leqslant n$ *then*

$$Ta=\sum_{i=1}^n\lambda_i\langle a,a_i'\rangle\, b_i$$

where $\|a_i'\|_{A'}=1$, $\|b_i\|_B\leqslant 1$, *and there exist* $b_i'\in B'$ *such that* $\langle b_i,b_j'\rangle=\delta_{ij}$ *and* $\|b_i'\|_{B'}\leqslant 1$.

12. (Peetre [19]). Denote by $\mathsf{A}_{\alpha\beta}(A,B)$ the set of (bounded linear) operators $T: A\to B$, A,B being Banach spaces, such that the induced operator $\tilde{T}: \lambda_\alpha(A)\to l_\beta(B)$ is bounded, where $l_\beta(B)=\{(b_n)_{n=1}^\infty \mid \sum_{n=1}^\infty \|b_n\|_B^\beta<\infty\}$ and $\lambda_\alpha(A)=\{(a_n)_{n=1}^\infty \mid \|\sum_{n=1}^\infty \varepsilon_n a_n\|_A\leqslant C$ for all $(\varepsilon_n)_{n=1}^\infty;\ \sum_{n=1}^\infty|\varepsilon_n|^{\alpha'}\leqslant 1,\ 1/\alpha+1/\alpha'=1\}$. Show that

$$T\in\mathsf{A}_{\alpha\beta}(A_0,B_0)\cap\mathsf{A}_{\alpha\beta}(A_1,B_1)$$

implies that

$$T\in\mathsf{A}_{\alpha\beta}(\bar{A}_{(\rho),\infty;K},\bar{B}_{(\sigma),\beta;K})$$

if $\bar{A}$ is quasi-linearizable (see Exercise 6, Chapter 3) and $\int_0^\infty(\sigma(t)/\rho(t))^\beta dt/t<\infty$. Here

$$\bar{A}_{(\rho),p;K}=\{a\in\Sigma(\bar{A})\mid(\int_0^\infty(K(t,a)/\rho(t))^p dt/t)^{1/p}<\infty\}$$

and ρ is positive function; $p=\infty$ has the usual meaning.

Hint: Note that if $\alpha>1$ then $\lambda_\alpha(A)=L(l_{\alpha'}, A)$, $1/\alpha+1/\alpha'=1$, and, interpolating $\tilde{T}$, show that $\tilde{T}: (\lambda_\alpha(A_0), \lambda_\alpha(A_1))_{(\rho),\infty;K} \to (l_\beta(B_0), l_\beta(B_1))_{(\rho),\infty;K} \subset (l_\beta(B_0), l_\beta(B_1))_{(\sigma),\beta;K} = l_\beta(\bar{B}_{(\sigma),\beta;K})$.

13. Consider the torus T. Show that the couple (C^0, C^1) is quasi-linearizable (see 3.13.6). Generalize to the semi-group case.

Hint: Put $V_0(t)f(x)=1/2t\int_{-t}^{t}(a(x)-a(x+h))\,dh$ and use the formula for $K(t,f;C^0,C^1)$ in Section 6.

14. (Bergh-Peetre [1]). Let V_p be defined as in Section 6. Show that

$$(V_{p_0}, V_{p_1})_{\theta,p} \subset V_p \quad (0<\theta<1)$$

if $1/p=(1-\theta)/p_0+\theta/p_1$ and $1/k \leqslant p_i \leqslant \infty$.

15. (Bergh-Peetre [1]). Prove that if f is an entire function of exponential type at most r and $f \in L_p$ $(1/k \leqslant p \leqslant 1)$ then

$$\|f\|_{V_p} \leqslant C r^{1/p} \|f\|_{L_p}.$$

Hint: (i) By Plancherel and Pólya (see Boas [1]) $\|f^{(k)}\|_{L_p} \leqslant C\|f\|_{L_p}$ $(r=1)$. (ii) For any discrete subset $X \subset R$, such that $|x-y| \geqslant 1$ if $x, y \in X$ and $x \neq y$, $(\sum_{x\in X}|f(x)|^p)^{1/p} \leqslant C\|f\|_{L_p}$. (iii) Split $\mathscr{I}$ into two families, one containing precisely those intervals I, for which $|I| \geqslant 1$.

16. (Bergh-Peetre [1]). Prove that if $1/k \leqslant p \leqslant \infty$ then

$$\dot{B}^{1/p}_{pp^*} \subset V_p \subset \dot{B}^{1/p}_{p\infty},$$

where $p^*=\min(1,p)$.

Hint: (i) Use the two previous exercises and interpolate. (ii) Use Exercise 6 in Chapter 6.

For the exercises 17—22 we give Brenner-Thomée-Wahlbin [1] as a general reference. In that work the reader will find a complete list of references.

17. Consider the one-dimensional heat-equation $\partial u/\partial t = \partial^2 u/\partial x^2$ with the initial value $u=f$ at $t=0$. Let $G_h(t)$ be a stable difference operator on L_∞ and assume that the corresponding operator P_h approximates $P=-D^2$ with order exactly s. Then we know from Theorem 7.4.2 that if

$$\|G_h(t)f - G(t)f\|_\infty \leqslant Ch^\sigma \quad \textit{uniformly in} \quad t=\lambda n h^2,$$

then $f \in B^\sigma_{\infty\infty}$. Now prove that if $\sigma>1$ and

$$\|G_h(t)f - G(t)f\|_\infty \leqslant Ch^\sigma, \quad t=\lambda n h^2 \ \text{fixed}$$

then $f \in B^{\sigma-1}_{\infty\infty}$.

Hint: Take $t=1$ and start with the estimate

$$\|\varphi_j * f\|_\infty \leqslant \sum_{2^{j-1}\leqslant |m| \leqslant 2^{j+1}} \|\varphi_j * \mathscr{F}^{-1} g_m * f\|_\infty,$$

where $g_m(\xi)=g(\xi-m)$ and $\sum_m g(\xi-m)=1$.

18. Let the assumptions of the previous exercise be satisfied and assume in addition that if $e(\xi)$ is the symbol of the difference scheme then $|e(\xi)|\leqslant \exp(-c\xi^2)$, where $c>0$. Prove that

$$\|G_h(t)f-G(t)f\|_\infty \leqslant Ct^{-1/2}h^\sigma \|f\|^\sigma_{B_{1\infty}}$$

if $1<\sigma\leqslant s$.

Hint: Note that $\|\mathscr{F}^{-1}a\hat{f}\|_\infty \leqslant C\|a\|_1\|f\|_1$. Use the proof of Theorem 7.4.2, but estimate the L_1-norm of $\exp(-tP_h(\xi))-\exp(-t\xi^2)$.

19. Consider the one-dimensional Schrödinger equation $\partial u/\partial t = i\partial^2 u/\partial x^2$ with the initial value $u=f$ at $t=0$. Let $G_h(t)$ be a stable difference scheme on L_2 and assume that the corresponding operator P_h approximates iD^2 with order s.

(a) Prove that

$$\|G_h(t)f-G(t)f\|_2 \leqslant C_T h^s \|f\|_2^{2+s}, \qquad t=\lambda nh^2 \leqslant T.$$

Deduce that for $0<\sigma<s$

$$\|G_h(t)f-G(t)f\|_2 \leqslant C_T h^\sigma \|f\|_{2\infty}^{\sigma(2+s)/s}, \qquad t\leqslant T.$$

(b) Prove that

$$\|G_h(t)f-G(t)f\|_\infty \leqslant C_T h^s \|f\|_{21}^{2+s+1/2}, \qquad t\leqslant T,$$

and for $0<\sigma<s$,

$$\|G_h(t)f-G(t)f\|_\infty \leqslant C_T h^\sigma \|f\|_{2\infty}^{\sigma(2+s)/s+1/2}, \qquad t\leqslant T.$$

20. Let Q_r be the set of all $\xi=(\xi_1,\dots,\xi_n)$ such that $|\xi_j|\leqslant r$ for $j=1,\dots,n$. Write

$$(Pf)_\alpha = P_\alpha(f) = \int_{Q_1} f(x+\alpha)dx,$$

$$(Ec)(x) = \sum_\alpha c_\alpha \chi(x-\alpha),$$

where χ is the characteristic function of Q_1.

(a) Writing T_a for the mapping $f\to\mathscr{F}^{-1}a\hat{f}$ prove that PT_aE is the mapping

$$(c_\alpha)_\alpha \to \left(\sum_\beta c_\beta P_{\alpha-\beta}(T_a\chi)\right)_\alpha.$$

(b) Prove that there is an infinitely differentiable function φ with compact support such that

$$\|\sum_\alpha b(\cdot+2\pi\alpha)\|_{M_p} \leqslant c\|b\|_{M_p} \leqslant \sup_{c\in l_p} \|(PT_{\varphi b}E)(c)\|_{l_p},$$

provided that b vanishes outside Q_π. Deduce that

$$\|\sum_\alpha b(\cdot+2\pi\alpha)\|_{M_p} \leqslant C\|b\|_{M_p}.$$

(c) Suppose that a is infinitely differentiable and 2π-periodic. Let η be infinitely differentiable and assume that $\eta(x)=1$ on Q_π, $\eta(x)=0$ outside $Q_{5\pi/4}$. Prove that

$$\|a\|_{M_p} \leqslant C\|\eta a\|_{M_p}.$$

21. Assume that e is the symbol of a difference operator and that $e(\xi)=\exp(i\theta(\xi))$ where θ is real, twice differentiable on $|\xi|<1$ and $\theta''(\xi)\neq 0$ for $|\xi|<1$. Use the previous exercise to prove that

$$\|e(\cdot)^n\|_{M_p} \leqslant Cn^{\tilde{p}}, \qquad \tilde{p}=|1/2-1/p|.$$

22. Consider the one-dimensional wave-equation $\partial u/\partial t=\partial u/\partial x$, $u=f$ at $t=0$. Let $G_h(t)$ be a difference operator with symbol $e(\xi)$ satisfying the assumptions of the previous exercise and assume that P_h approximates D with order s. Prove that

$$\|G_h(t)f-G(t)f\|_p \leqslant Ch^{q(\sigma)}\|f\|_{p\infty}^\sigma, \qquad t=\lambda nh\leqslant T,$$

where

$$q(\sigma) = \begin{cases} \sigma-\tilde{p} & \text{if } 0<\sigma<(s+1)\tilde{p}, \\ \sigma s/(s+1) & \text{if } (s+1)\tilde{p}<\sigma<s+1 \end{cases}$$

and $0<\sigma<s+1$.

23. (Löfström-Thomée [1], Peetre [17]). Prove, using induction on N, that

$$\|D^\gamma w\|_q \leqslant c\|w\|_\infty^{1-\theta}(\|w\|_2^N)^\theta,$$

if $0<\theta=|\gamma|/N\leqslant 1$, $q=2/\theta$. (The norms are the norms in $L_q(\mathbb{R}^n)$, $L_\infty(\mathbb{R}^n)$ and $H_2^N(\mathbb{R}^n)$ respectively.) Use this result to show that

$$\|D^\alpha f(w)\|_2 \leqslant C_f(\|w\|_\infty+1)^{N-1}(\|w\|_2^N+1),$$

if $|\alpha|=N$, $w\in L_\infty\cap H_2^N$ and $f\in\mathscr{S}$. Show that if in addition $f(0)=0$, then

$$w\in B_{21}^s \Rightarrow f(w)\in B_{21}^s, \qquad (s\geqslant n/2).$$

(Cf. Exercise 9 and 10 of Chapter 3.)

24. Consider the non-linear initial value problem

$$\begin{cases} \partial u/\partial t = \partial u/\partial x + \rho u^{r+1}, & t>0, \quad x\in \mathbb{R}, \\ u(0,x)=w(x), & x\in \mathbb{R} \end{cases}$$

where ρ is a non-vanishing constant and r a positive integer. Prove that for a given $w\in L_\infty = L_\infty(\mathbb{R})$ there is a unique solution $u(t,x)$, defined and bounded on $0\leqslant t\leqslant T$, $x\in\mathbb{R}$ for some T. Find the upper bound for T, in the three cases

$$\text{(i) } \rho>0, \quad \text{(ii) } \rho<0, \quad r \ \textit{odd}, \quad \text{(iii) } \rho<0, \quad r \ \textit{even}.$$

Finally prove that if $w\in B^s_{21}$ then $u(t,\cdot)\in B^s_{21}$ for $s\geqslant 1/2$.

7.6. Notes and Comment

The possibility of applying interpolation techniques, as we have described them, to approximation theory was indicated by Peetre [10] in 1963. Since then, interpolation has been used in connection with, e.g., approximation by rational functions and by spline functions, trigonometric approximation, approximation by eigenfunction expansions in general, moment problems and the other topics treated in the previous sections of this chapter. (See, e.g., the works of P.L. Butzer and his coauthors mentioned below.)

Approximation by spline functions and by rational functions are closely related. See Peetre-Sparr [1] and Peetre [23]. *Approximation by spline functions* is considered also by Bergh-Peetre [1] in connection with spaces V_p $(0<p\leqslant\infty)$ of functions of bounded variation on the real line. (See also Brudnyi [2].) More precisely, given a fixed integer $k\geqslant 1$ with $k^{-1}\leqslant p\leqslant\infty$, V_p is the linear spaces of all measurable, locally bounded functions on $\mathbb{R}$, such that, for every family $\mathscr{I}=\{I\}$ of disjoint intervals $I=(a,b)$,

$$(\textstyle\sum_{I\in\mathscr{I}}(\inf_{\pi\in\mathscr{P}}\sup_{x\in I}|f(x)-\pi(x)|)^p)^{1/p}\leqslant C,$$

where C is independent of $\mathscr{I}$ and $\mathscr{P}=\mathscr{P}_k$ denotes the space of all polynomial functions on R of degree at most $k-1$. The (quasi-) norm on V_p is the supremum of the right hand side over all families $\mathscr{I}$. The approximation result is the following: *Let f belong to the closure of $\mathscr{D}$ (infinitely differentiable functions with compact support) in supremum norm. Then*

$$\inf_{g\in \mathrm{Spl}(N)}\sup_{x\in\mathbb{R}}|f(x)-g(x)| = O(N^{-1/p}) \qquad (N\to+\infty)$$

iff

$$f\in(V^0_{1/k}, C^0)_{\theta,\infty} \qquad (1/p=(1-\theta)k),$$

where $\mathrm{Spl}(N)$ *consists of functions with compact support such that*

$$f^{(k+1)}(x)=\textstyle\sum_{j=1}^N A_j\delta(x-a_j) \qquad (\text{in } \mathscr{D}'),$$

and the superscript ^{0} *signifies "the closure of* $\mathscr{D}$*" in the respective norms.* In our notation, this result could be rewritten as:

$$(\mathrm{Spl}, C^0)_{1/p,\infty;E} = (V^0_{1/k}, C^0)_{\theta,\infty;K},$$

provided that $1/p=(1-\theta)k$. The relation between the spaces V_p and the spaces $\dot{B}^s_{pq}$ is the subject of Exercises 14—16. The proof of the approximation result is based on two inequalities of Jackson and Bernstein type. Once these inequalities are established, it only remains to characterize the space $(V^0_{1/k}, C^0)_{\theta,\infty;K}$. This is done via a formula for the K-functional. (Cf. the proof of Theorem 7.2.4.)

Consider now *the couple* (C^0, C^1), where C^0 is the space of bounded real-valued uniformly continuous functions on the real line $\mathbb{R}$, and C^1 is the subspace containing those which have their first derivative in C^0. The (semi-) norms are

$$\|f\|_{C^0} = \sup_{\mathbb{R}} |f(x)|,$$

$$\|f\|_{C^1} = \sup_{\mathbb{R}} |Df(x)|$$

respectively. Peetre [14] has shown that

$$K(t,f;C^0,C^1) = \tfrac{1}{2}\omega^*(2t,f),$$

where ω is the modulus of continuity of f:

$$\omega(t,f) = \sup_{x\in\mathbb{R}} \sup_{|h|<t} |f(x+h)-f(x)|,$$

and ω^* is the least concave majorant of ω. From this formula it follows that

$$\sup_t K(t,f)/\varphi(t) = \sup_t \omega(t,f)/2\varphi(t/2),$$

where φ is positive and concave. We may interprete the last formula as saying that any $\mathrm{Lip}(\varphi(\cdot))$ space is a K-interpolation space with respect to the couple (C^0, C^1) if φ is concave. Conversely, Bergh [1] has shown that if $\mathrm{Lip}(\varphi(\cdot))$ is an interpolation space with respect to the couple (C^0, C^1) then φ is essentially concave (see Bergh [1] for precise statements). The formula for $K(t,f;C^0,C^1)$ may also be seen from the point of view of approximation theory. Consider now the torus $\mathbb{T}$ instead of the real line $\mathbb{R}$. The same formula for $K(t,f)$ holds in this case. Kornejčuk [1] has shown that (φ concave)

$$E(n-1,f;C^0,T) \leqslant \tfrac{1}{2}\varphi(\pi/n)$$

iff

$$f \in \mathrm{Lip}(\varphi(\cdot)),$$

where T is the space of trigonometric polynomials and $\|f\|_T$ is the degree of f. The connection between E and K is provided by the inequality

$$E(n-1,f;C^0,T) \leqslant K(\pi/2n,f;C^0,C^1).$$

(See Peetre [14] for details.)

A formula for the functional $K(t, a; C^0, C^2)$ has recently been found by J. Friberg, who also has established related approximation results.

7.6.1.—2. The exposition of these sections follows closely that of Peetre-Sparr [1].

As we stated in Chapter 1, the classical results by Jackson [1] and Bernstein [1] from 1912, corresponding to (1) and (2) of Section 2, were given for the torus $\mathbb{T}$ and supremum norms. Cf., e.g., Lorentz [3].

Theorem 7.2.4, a consequence of (1) and (2), stated for $\mathbb{R}^n$ also holds, mutatis mutandis, for the torus $\mathbb{T}^n$. This is proved in quite a similar way (see Exercise 6, 7). Partially corresponding results hold when $0<p<1$ (cf. Peetre [23]). Note that Theorem 7.2.4 gives yet another possible way of defining the spaces B^s_{pq}, at least for $s>0$ and $1 \leqslant p \leqslant \infty$.

Many applications of interpolation theory to approximation theory can be found in the book of Butzer-Behrens [1], which also contains a large list of references. (See also Butzer-Nessel [1].) There are several conference proceedings with applications to approximation theory and harmonic analysis (and with valuable lists of references), for instance Butzer-Nagy [1] (see notably the articles by Bennett [3], Gilbert [2] and Sagher [4]), Butzer-Nagy [2], Butzer-Kahane-Nagy [1]. See also Alexits-Stečkin [1] where a survey article by Peetre [25] can be found.) Other applications to approximation theory and harmonic analysis are given in Peetre [3, 4, 8, 11, 22, 23], Löfström-Peetre [1], Peetre-Vretare [1], Löfström [1,2,3], Sagher [4], Varopoulos [1], Hedstrom-Varga [1]. (See also the exercises.)

7.6.3. There are several papers concerning the interpolation of ideals of operators. Apart from the work of Peetre-Sparr [1] (and references given there) we mention here the works of Pietsch [1,2], Pietsch-Triebel [1], Triebel [2], Peetre [24], Merucci-Pham the Lai [1], Favini [1], Gilbert [2]. (See also Gohberg-Krein [1] and Peetre-Sparr [2].)

7.6.4. This section is taken over from Löfström [2]. Further results are given in the exercises.

An important inspiration for interpolation theory is the theory of partial differential equations. Conversely interpolation theory has been applied to partial differential equations. We mention here the books of Lions-Magenes [2]. (See also Lions [4], Tartar [1], Triebel [1, 3].) Applications to numerical integration of partial differential equations has been given by several authors. We refer the reader to the lecture notes by Brenner-Thomée-Wahlbin [1] and the references given there.

Interpolation theory has been applied to the theory of non-integer powers of operators. See Komatsu [1] for a systematic treatment. Other related papers (using interpolation theory) are Yoshikawa [2], Yoshinaga [1].

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List of Symbols

General notation

The complex interpolation method

The real interpolation method

Operators and functionals

Spaces of distributions

Lebesgue-spaces

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